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TECHNICAL NOTE 4186

HEAT TRANSFER IN ISOTROPIC TURBULENCE DURING THE FINAL PERIOD OF DECAY

By D. W. Dunn and W. H. Reid

The Johns Hopkins University
and
Ballistic Research Laboratories, U. S. Army

Washington
June 1958

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HEAT TRANSFER IN ISOTROPIC TURBULENCE

DURING THE FINAL PERIOD OF DECAY

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SUMMARY

The problem of heat transfer in isotropic turbulence with a constant mean temperature gradient is considered during the final period of decay. The Reynolds and Péclet numbers are then very small, and all triple correlation terms can be neglected in the equations for the double correlations. On this basis, it is found that the temperature field ultimately becomes independent of the initial conditions on the temperature and has characteristics determined only by the mean temperature gradient, the physical properties of the fluid, and the characteristics of the turbulence. Detailed analytical and numerical results are obtained for the asymptotic state.

The mean turbulent heat transfer is in the direction of the mean temperature gradient, with a magnitude proportional to the magnitude of the latter. Although it approaches zero when the Prandtl number approaches zero, its dependence on the Prandtl number is not large for Prandtl numbers of order unity and larger. This type of Prandtl number dependence is typical for many of the other results depending on both velocity and temperature fluctuations. In fact, the rate of decrease with separation distance of the two-point temperature-velocity correlation varies little over the full range of Prandtl numbers and is always about the same as it is for the double velocity correlation. In contrast, all results involving only temperature fluctuations display a strong dependence on the Prandtl number. For example, for small Prandtl numbers the double temperature correlation falls off much more slowly with separation distance than the velocity correlation does, while for large Prandtl numbers the opposite is true.

INTRODUCTION

The simplest case of turbulent heat transfer is the problem first considered by Corrsin (ref. 1), in which the temperature of the fluid is specified to have a constant mean gradient in some preferred direction, while the velocity field, which is assumed independent of the temperature

field, is regarded as isotropic and known. Thus, though mean values associated with the velocity field only are isotropic, those associated with the temperature field are axisymmetric. A study of this problem, although it is highly idealized, is expected to give some idea of the nature of turbulent heat transfer and, by analogy, also of turbulent mass transfer.

Such a study is also a natural preliminary step before attempting the much more complicated problem of turbulent momentum transfer. The simplest example of the latter, the so-called homogeneous shear flow with a constant mean velocity gradient studied by Reiss (ref. 2) and later by Burgers and Mitchner (ref. 3), is mathematically similar to the present problem in some respects, though with additional complications due to the vector nature of the transported quantity. The present problem, because of the scalar nature of the transported quantity and the resulting more restrictive symmetry conditions, is considerably simpler mathematically. Its solution may be expected to provide useful clues towards the solution of the more difficult problem of homogeneous shear flow.

In addition to being of theoretical interest, this idealized problem of turbulent heat transfer is of considerable practical interest, since a direct experimental investigation of it appears to be feasible. For example, if the horizontal bars of the usual turbulence-producing grid in a wind tunnel are heated with a proper variation in the temperatures of successive bars, the conditions of the idealized problem should be reproduced fairly closely in a region downstream of the grid. There are, of course, various approximations involved in relating the experimental situation to the theoretical problem, but they appear to be no more serious than the ones normally associated with the wind-tunnel study of isotropic turbulence.

Although the problem as stated appears to be the simplest problem of turbulent transfer that can be formulated, there are still formidable mathematical difficulties present in the general case, when the inertial transfer terms are dominant in the basic equations. In particular, the system of equations for temperature mean values that can be derived from the heat-transfer equation is incomplete, just as is the well-known system for velocity mean values derived from the dynamical equations. Of course, as in the latter case, one could, to obtain a determinate system, make some type of assumption such as the one most commonly made at the present time that fourth-order mean values are related to second-order ones as they would be for a normal probability distribution.

Proudman and Reid (ref. 4) have recently used this statistical assumption in an analysis of isotropic turbulence involving mean values of velocities at several points but at the same time (see also ref. 5 by Tatsumi). Reid (ref. 6) has applied the idea in a similar manner to the study of the convective effects of isotropic turbulence on arbitrary

vector and scalar fields, including, in particular, the case of isotropic temperature fluctuations in isotropic turbulence. Chandrasekhar (ref. 7) has used it in an alternative approach to isotropic turbulence, in which mean values at different times as well as at different points are considered. At the present time, however, the possibilities of the assumption have not been fully exploited even in these problems involving complete isotropy. Until further progress has been made with the latter, it does not appear worthwhile as yet to pursue this approach further in the present more complicated problem.

The objectives of the present study, as well as of the earlier investigations of Corrsin (ref. 1) and Mélèse (ref. 8), are considerably more limited, being restricted to results that can be derived without a detailed consideration of the triple and higher order correlations. Corrsin and Mélèse have obtained in this manner results of a general nature for a variety of different situations but have not treated any special case in detail. In the present investigation, attention will be restricted mainly to the final period of decay, and for this case essentially complete results will be derived for the double temperature correlation and the double temperature-velocity correlation and their Fourier transforms. During the final period, the turbulence Reynolds numbers are so low that inertial transfer effects are very small, and the triple-correlation terms in the equations for the double correlations can be neglected. To this order of approximation the problem is determinate, and the statistical hypothesis mentioned above is not required. The conclusions may be expected to give an approximate description of the decay of the velocity and temperature fluctuations far downstream of the grid in the experiment described earlier.

Although some of the most interesting features of turbulent heat and momentum transport are obviously absent during the final period of decay, the latter must still be recognized as an important aspect of the general problem. For this reason, and because of the relative simplicity of the analysis, a detailed treatment of this special case is appropriate first, before proceeding with the more difficult general case.

This research was carried out at the Department of Aeronautics, The Johns Hopkins University, under the sponsorship and with the financial assistance of the National Advisory Committee for Aeronautics, and at the Ballistic Research Laboratories, U. S. Army, Aberdeen Proving Ground. The authors would like to thank Prof. Stanley Corrsin for many helpful discussions. Thanks are also due to Mrs. Frances Chenoweth and Mrs. Barbara Jacobs for assistance with the numerical calculations.

SYMBOLS

$E(k)$	defining scalar of $\Phi_{ij}(\underset{\sim}{k})$; isotropic energy spectrum function
$f(r)$	defining scalar of $R_{ij}(\underset{\sim}{r})$; longitudinal velocity correlation coefficient
$\underset{\sim}{G}$	vector with components, $G_i = \dfrac{\partial\vartheta}{\partial x_i}$
$\underset{\sim}{k}$	wave-number vector
$L(r,m)$	defining scalar of $L_i(\underset{\sim}{r},\underset{\sim}{\lambda})$
$L'(r,m)$	defining scalar of $L_i'(\underset{\sim}{r},\underset{\sim}{\lambda})$

$$L_i(\underset{\sim}{r},\underset{\sim}{\lambda}) = \frac{1}{2}\left[\overline{\vartheta(\underset{\sim}{x},t)u_i(\underset{\sim}{x} + \underset{\sim}{r},t)} + \overline{\vartheta(\underset{\sim}{x} + \underset{\sim}{r},t)u_i(\underset{\sim}{x},t)}\right] \quad \text{(axisymmetric)}$$

$$L_i'(\underset{\sim}{r},\underset{\sim}{\lambda}) = \frac{1}{2}\left[\overline{\vartheta(\underset{\sim}{x},t)u_i(\underset{\sim}{x} + \underset{\sim}{r},t)} - \overline{\vartheta(\underset{\sim}{x} + \underset{\sim}{r},t)u_i(\underset{\sim}{x},t)}\right] \quad \text{(axisymmetric)}$$

L_ϑ	thermal integral scale (see eq. (87))
$M(r,m)$	temperature correlation coefficient (see eq. (96))

$$m = \frac{(\underset{\sim}{r}\cdot\underset{\sim}{\lambda})}{r} = \cos\alpha$$

$N(r,m)$	temperature-velocity correlation coefficient (see eq. (111))
P_{λ_ϑ}	Péclet number, $\left(\overline{u^2}\right)^{1/2}\lambda_\vartheta/\kappa$
p	pressure
R_h	heat-transfer correlation coefficient (see eq. (77))

$$R_{ij}(\underset{\sim}{r}) = \overline{u_i(\underset{\sim}{x},t)u_j(\underset{\sim}{x} + \underset{\sim}{r},t)} \quad \text{(isotropic)}$$

R_λ	turbulence Reynolds number, $\left(\overline{u^2}\right)^{1/2}\lambda/\nu$

$\underset{\sim}{r}$	separation vector in correlation functions
T	temperature
t	time
t_o	initial time
u_n	component of $\underset{\sim}{u}$ in a direction normal to $\underset{\sim}{\lambda}$
u_p	component of $\underset{\sim}{u}$ parallel to $\underset{\sim}{\lambda}$, $u_i\lambda_i$
$\overline{u^2}$	velocity vector with components $u_i(\underset{\sim}{x},t)$
$\underset{\sim}{u} = u_i(\underset{\sim}{x},t)$	
x_i	space coordinate
α	angle between vector $\underset{\sim}{r}$ and unit vector $\underset{\sim}{\lambda}$
β	magnitude of mean temperature gradient (a positive constant), $\lvert\nabla\overline{T}\rvert$
γ	angle between vector $\underset{\sim}{k}$ and unit vector $\underset{\sim}{\lambda}$
δ_{ij}	Kronecker delta
ϵ_{jkl}	permutation symbol
$\dfrac{\Theta(\underset{\sim}{k},\underset{\sim}{\lambda})}{2\pi k^2} \equiv \dfrac{\Theta(k,\mu)}{2\pi k^2}$	Fourier transform of $\theta(\underset{\sim}{r},\underset{\sim}{\lambda})$ (axisymmetric)
$\hat{\Theta}(k)$	three-dimensional temperature spectrum function, $\int_{-1}^{1} \Theta(k,\mu)\, d\mu$
$\theta(\underset{\sim}{r},\underset{\sim}{\lambda}) \equiv \theta(r,m)$	$\overline{\vartheta(\underset{\sim}{x},t)\vartheta(\underset{\sim}{x}+\underset{\sim}{r},t)}$ (axisymmetric)

$\vartheta(\underset{\sim}{x},t)$	deviation of temperature from its local mean value
κ	thermal diffusivity
$\Lambda(k,\mu)$	defining scalar of $\Lambda_i(\underset{\sim}{k},\underset{\sim}{\lambda})$
$\Lambda'(k,\mu)$	defining scalar of $\Lambda_i'(\underset{\sim}{k},\underset{\sim}{\lambda})$
$\Lambda_i(\underset{\sim}{k},\underset{\sim}{\lambda})$	Fourier transform of $L_i(\underset{\sim}{r},\underset{\sim}{\lambda})$ (axisymmetric)
$\Lambda_i'(\underset{\sim}{k},\underset{\sim}{\lambda})$	Fourier transform of $L_i'(\underset{\sim}{r},\underset{\sim}{\lambda})$ (axisymmetric)
$\hat{\Lambda}(k)$	three-dimensional heat-transfer spectrum function, $-2\pi k^2 \int_{-1}^{1} \Lambda_p(k,\mu)\, d\mu$
λ	turbulence microscale (see eq. (67))
$\lambda_i = \dfrac{\left(\partial\overline{T}/\partial x_i\right)}{\left\lvert\nabla\overline{T}\right\rvert}$	
λ_ϑ	thermal microscale (see eq. (82))
$\underset{\sim}{\lambda}$	unit vector in direction of mean temperature gradient
$\mu = \dfrac{\left(\underset{\sim}{k}\cdot\underset{\sim}{\lambda}\right)}{k} = \cos\gamma$	
ν	kinematic viscosity
ρ	density
σ	Prandtl number, ν/κ
$\Phi_{ij}(\underset{\sim}{k})$	Fourier transform of $R_{ij}(\underset{\sim}{r})$ (isotropic)
$\underset{\sim}{\omega}$	vorticity vector with components $\omega_j = \epsilon_{jkl}\dfrac{\partial u_l}{\partial x_k}$

Subscripts:

i,j,k,l	tensor indices
m	maximum
n	normal
o	at time t_o
p	parallel

FORMULATION OF PROBLEM

For the problem under consideration, it is assumed that the velocity field is isotropic and unaffected by temperature variations. The governing equations for the velocity and pressure are then the Navier-Stokes equation

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu\nabla^2 u_i \tag{1}$$

where ∇^2 is the Laplacian operator $\frac{\partial^2}{\partial x_j\,\partial x_j}$, and the continuity equation

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{2}$$

The quantities ρ and ν are the density and kinematic viscosity, respectively, and are assumed to be constant. It is supposed that in the fluid a constant mean temperature gradient β is maintained in the direction of a unit vector $\underset{\sim}{\lambda}$ by some external agency $\left(\text{i.e., } \beta = \left|\nabla\overline{T}\right| \text{ and } \lambda_i = \frac{\partial\overline{T}/\partial x_i}{\left|\nabla\overline{T}\right|}\right)$, so that

$$T\left(\underset{\sim}{x},t\right) = T_o + \beta\lambda_i x_i + \vartheta\left(\underset{\sim}{x},t\right) \tag{3}$$

where T_o is a constant and $\vartheta\left(\underset{\sim}{x},t\right)$ denotes the deviation of the temperature from its local mean value. From the heat-transfer equation

$$\frac{\partial T}{\partial t} + u_j \frac{\partial T}{\partial x_j} = \kappa\nabla^2 T \tag{4}$$

the governing equation for the temperature fluctuations is obtained as

$$\frac{\partial\vartheta}{\partial t} + u_j \frac{\partial\vartheta}{\partial x_j} + \beta\lambda_j u_j = \kappa\nabla^2\vartheta \tag{5}$$

where κ is the thermal diffusivity, also assumed to be constant. The equation satisfied by the mean temperature, which is obtained by averaging equation (4), reduces in this problem to the obvious result that $\frac{\partial\overline{\vartheta u_j}}{\partial x_j} = 0$, that is, that the mean turbulent heat transfer vector is a constant. The temperature field, because of the preferred direction λ, is seen to be axisymmetric rather than isotropic.

The preceding formulation evidently also applies to turbulent mass transfer, if the temperature T is replaced by concentration (assumed small), and the thermal diffusivity κ is replaced by the diffusion coefficient.

The various physical restrictions, such as restrictions on the magnitudes of the velocity and temperature variations, which are required in order for the above formulation to give an adequate approximation to reality, are well known (see Corrsin's comments in refs. 1 and 9, e.g.).

During the final period of decay when the turbulence Reynolds number is small, molecular transfer effects are expected to dominate inertial transfer effects. Quantitative estimates of their relative importance can be obtained by order-of-magnitude considerations. For example, in the dynamical equation (1) the viscous terms $\nu\nabla^2 u_i$ are in this case among the dominant terms, and the inertial transfer terms $u_j\, \partial u_i/\partial x_j$ and pressure term $-(1/\rho)(\partial p/\partial x_i)$ are of relative order of magnitude $\left(\overline{u^2}\right)^{1/2}\lambda/\nu$, where λ is the microscale. The quantities $\left(\overline{u^2}\right)^{1/2}$ and λ evidently represent a significant velocity and a significant length, respectively, for the energy-containing eddies of the turbulence as well as for the dissipative eddies under these circumstances (i.e., the microscale is now of the same order of magnitude as the integral scale). Similarly, in the heat-transfer equation (5) the thermal-conduction terms $\kappa\nabla^2\vartheta$ are among the dominant terms, and the inertial transfer terms $u_j\, \partial\vartheta/\partial x_j$ are of relative order of magnitude $\sigma\left(\overline{u^2}\right)^{1/2}\lambda_\vartheta/\nu$, where σ is the Prandtl number and λ_ϑ is some length characteristic of the temperature fluctuations. The thermal microscale, defined by Corrsin (ref. 9) for the temperature field in a manner analogous to the definition of λ for the velocity field, is evidently a significant choice for the length λ_ϑ.

Hence, the final period of decay is characterized by the inertial transfer terms in equations (1) and (5) being negligible or more precisely by the satisfaction of the following conditions:

$$\frac{\left(\overline{u^2}\right)^{1/2}\lambda}{\nu} \ll 1 \tag{6}$$

and

$$\frac{\sigma\left(\overline{u^2}\right)^{1/2}\lambda_\vartheta}{\nu} \ll 1 \tag{7}$$

The first condition implies that the Reynolds number must be small, as was to be expected, and the second implies that the Péclet number must also be small. When the Prandtl number is not too large, it is reasonable to expect that λ_ϑ and λ are of the same order of magnitude, so that the Péclet number is of the order of the Reynolds number and the two conditions are essentially equivalent.

In all subsequent considerations, the approximations described above will be accepted, and the dynamical equation and heat-transfer equation will be used in the approximate forms

$$\frac{\partial u_i}{\partial t} = \nu\nabla^2 u_i \tag{8}$$

and

$$\frac{\partial\vartheta}{\partial t} + \beta\lambda_j u_j = \kappa\nabla^2\vartheta \tag{9}$$

The relative errors in these equations are $O\left(R_\lambda\right)$ in the first and $O\left(P_{\lambda_\vartheta}\right)$ in the second, where R_λ and P_{λ_ϑ} are the Reynolds number and Péclet number defined in relations (6) and (7), respectively.

The situation under consideration, then, is as follows. At some time t_o after the initial instant of generation of the velocity and temperature fields, the turbulence intensity has decreased to such an extent that the Reynolds and Péclet numbers are both very small and equations (8) and (9) are satisfactory approximations. With their use, the subsequent variation of the velocity and temperature fluctuations can then be determined to the same order of approximation in terms of conditions at time t_o.

In general, of course, the relations between conditions at time t_o and those at the initial instant of generation of the velocity-temperature field are beyond the scope of this approximate analysis. These relations depend in a complicated and unknown manner on the neglected terms in equations (8) and (9), which are initially important if the initial turbulence level is high enough. For present purposes, the conditions at time t_o are assumed to be known, at least in a statistical sense. The considerations will be limited mainly to the double temperature correlation $\overline{\vartheta\vartheta'}$ and the temperature-velocity correlation $\overline{\vartheta u_i'}$. The results to be presented for the double velocity correlation $\overline{u_i u_j'}$ are well known, but they will be included for the sake of completeness.

Solutions of Equations for Fourier Transforms of Correlation Functions

Let the correlation functions be defined as

$$\theta(\underset{\sim}{r},\underset{\sim}{\lambda}) = \overline{\vartheta(\underset{\sim}{x})\vartheta(\underset{\sim}{x} + \underset{\sim}{r})} \tag{10}$$

$$L_i(\underset{\sim}{r},\underset{\sim}{\lambda}) = \frac{1}{2}\left[\overline{\vartheta(\underset{\sim}{x})u_i(\underset{\sim}{x} + \underset{\sim}{r})} + \overline{\vartheta(\underset{\sim}{x} + \underset{\sim}{r})u_i(\underset{\sim}{x})}\right] \tag{11}$$

$$L_i'(\underset{\sim}{r},\underset{\sim}{\lambda}) = \frac{1}{2}\left[\overline{\vartheta(\underset{\sim}{x})u_i(\underset{\sim}{x} + \underset{\sim}{r})} - \overline{\vartheta(\underset{\sim}{x} + \underset{\sim}{r})u_i(\underset{\sim}{x})}\right] \tag{12}$$

where the dependence on $\underset{\sim}{\lambda}$ has been indicated explicitly to emphasize the axisymmetric character of these mean values. For the sake of brevity, the dependence on the time will not be indicated. It is to be noted that mean values as well as fluctuating quantities are time dependent, since a decaying velocity-temperature field is being considered. With the aid of equations (8) and (9), the correlation functions, equations (10) to (12), are seen to satisfy the equations

$$\frac{\partial\theta}{\partial t} + 2\beta\lambda_i L_i = 2\kappa\nabla^2\theta \tag{13}$$

$$\frac{\partial L_i}{\partial t} + \beta\lambda_j R_{ij} = (\kappa + \nu)\nabla^2 L_i \tag{14}$$

$$\frac{\partial L_i{}'}{\partial t} = (\kappa + \nu)\nabla^2 L_i{}' \tag{15}$$

where $R_{ij}(\underset{\sim}{r}) = \overline{u_i(\underset{\sim}{x})u_j(\underset{\sim}{x} + \underset{\sim}{r})}$ is the double-velocity-correlation tensor usually denoted by this symbol, which to the present order of approximation satisfies the equation

$$\frac{\partial R_{ij}}{\partial t} = 2\nu\nabla^2 R_{ij} \tag{16}$$

Equations (13) to (15) are special cases of the more general equations derived by Chandrasekhar (ref. 10).

Since the velocity field is assumed to be isotropic, $R_{ij}(\underset{\sim}{r})$ is an isotropic tensor, solenoidal in both of its indices. It can therefore be represented in the usual way in terms of the longitudinal velocity-correlation function $\overline{u^2}f(r)$, where $\overline{u^2}$ denotes the mean-square value of one velocity component. On the other hand, the vector $L_i(\underset{\sim}{r},\underset{\sim}{\lambda})$ is axisymmetric, so that in general it requires two defining scalars and can be expressed in the form

$$L_i(\underset{\sim}{r},\underset{\sim}{\lambda}) = L_1 r_i + L_2\lambda_i \tag{17}$$

where L_1 and L_2 are arbitrary functions of

$$\left.\begin{aligned} \underset{\sim}{r}\cdot\underset{\sim}{r} &= r^2 \\ \underset{\sim}{r}\cdot\underset{\sim}{\lambda} &= rm \end{aligned}\right\} \tag{18}$$

When $L_i(\underset{\sim}{r},\underset{\sim}{\lambda})$ is solenoidal, as it is here, it requires only one defining scalar $L = L(r,m)$, which, as Chandrasekhar (ref. 11) has shown, is related to L_1 and L_2 in the manner

$$L_1 = -\left(rmD_r + D_m\right)L \tag{19}$$

$$L_2 = \left(r^2D_r + rmD_m + 2\right)L \tag{20}$$

where the differential operators D_r and D_m have the meaning

$$\left.\begin{aligned} D_r &= \frac{1}{r}\frac{\partial}{\partial r} - \frac{m^2}{r^2}\frac{\partial}{\partial m} \\ D_m &= \frac{1}{r}\frac{\partial}{\partial m} \end{aligned}\right\} \tag{21}$$

Exactly analogous considerations apply to the vector L_i' and its defining scalars L_1' and L_2', which can be expressed in terms of a single scalar L' as described by relations (19) and (20).

Equations (13) to (16) could be reduced to equations for the defining scalars described above, if desired. This is the procedure followed by Chandrasekhar (ref. 10). In practice, however, the desired information can most easily be derived by considering the Fourier transforms of these equations.

Thus, the Fourier transform of $\theta(\underset{\sim}{r},\underset{\sim}{\lambda})$ is defined in the form

$$\frac{\Theta(k,\mu)}{2\pi k^2} = (2\pi)^{-3}\int \theta(\underset{\sim}{r},\underset{\sim}{\lambda})e^{-i\underset{\sim}{k}\cdot\underset{\sim}{r}}\, d\underset{\sim}{r} \tag{22}$$

where $\underset{\sim}{k}\cdot\underset{\sim}{\lambda} = k\mu$ and where $\Theta(k,\mu)$ has been so normalized that

$$\overline{\vartheta^2} = \int_0^\infty \int_{-1}^1 \Theta(k,\mu)\ d\mu\ dk \qquad (23)$$

Likewise, the Fourier transforms of $L_i(\underset{\sim}{r},\underset{\sim}{\lambda})$ and $L_i'(\underset{\sim}{r},\underset{\sim}{\lambda})$ can be defined in the manner

$$\Lambda_i(\underset{\sim}{k},\underset{\sim}{\lambda}) = (2\pi)^{-3}\int L_i(\underset{\sim}{r},\underset{\sim}{\lambda})e^{-i\underset{\sim}{k}\cdot\underset{\sim}{r}}\ d\underset{\sim}{r} \qquad (24)$$

with a similar relation for the transform Λ_i' of L_i'. Finally, equations (13) to (16) can be transformed to

$$\frac{\partial\Theta}{\partial t} + 4\pi\beta k^2\lambda_i\Lambda_i = -2\kappa k^2\Theta \qquad (25)$$

$$\frac{\partial\Lambda_i}{\partial t} + \beta\lambda_j\Phi_{ij} = -(\kappa + \nu)k^2\Lambda_i \qquad (26)$$

$$\frac{\partial\Lambda_i'}{\partial t} = -(\kappa + \nu)k^2\Lambda_i' \qquad (27)$$

and

$$\frac{\partial\Phi_{ij}}{\partial t} = -2\nu k^2\Phi_{ij} \qquad (28)$$

where $\Phi_{ij}(\underset{\sim}{k})$ is the Fourier transform of $R_{ij}(\underset{\sim}{r})$ in the sense of equation (24).

Since $\Phi_{ij}(\underset{\sim}{k})$ is an isotropic tensor it can be written in the well-known form

$$\Phi_{ij}(\underset{\sim}{k}) = \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right)\frac{E(k)}{4\pi k^2} \tag{29}$$

where $E(k)$ is the isotropic energy spectrum function. The representation of the orthogonal $\left(\text{i.e., } k_i\Lambda_i = 0\text{, corresponding to } \frac{\partial L_i}{\partial r_i} = 0\right)$ axisymmetric vector $\Lambda_i\left(\underset{\sim}{k},\underset{\sim}{\lambda}\right)$ in terms of a single defining scalar can, by the technique of Proudman and Reid (ref. 4), be written directly in the form

$$\Lambda_i\left(\underset{\sim}{k},\underset{\sim}{\lambda}\right) = \left(\delta_{i\alpha} - \frac{k_i k_\alpha}{k^2}\right)\Lambda(k,\mu)\lambda_\alpha \tag{30}$$

where Λ is a function of $\underset{\sim}{k}\cdot\underset{\sim}{k} = k^2$ and $\underset{\sim}{k}\cdot\underset{\sim}{\lambda} = k\mu$. Similarly, there is an analogous expression for the orthogonal axisymmetric vector Λ_i' in terms of a single scalar function Λ'. Thus, the scalar forms of equations (25) to (28) become

$$\frac{\partial\Theta}{\partial t} + 2\kappa k^2\Theta = -4\pi\beta\left(1 - \mu^2\right)k^2\Lambda \tag{31}$$

$$\frac{\partial\Lambda}{\partial t} + (\kappa + \nu)k^2\Lambda = -\beta\,\frac{E}{4\pi k^2} \tag{32}$$

$$\frac{\partial\Lambda'}{\partial t} + (\kappa + \nu)k^2\Lambda' = 0 \tag{33}$$

$$\frac{\partial E}{\partial t} + 2\nu k^2 E = 0 \tag{34}$$

The derivation of the solutions of these equations satisfying given initial conditions at time t_o is straightforward, and the results are as follow:

$$E(k,t) \doteq E_o(k)e^{-2\nu k^2(t-t_o)} \tag{35}$$

$$\Lambda(k,\mu,t) = -\frac{\beta E_o(k)}{4\pi(\kappa - \nu)k^4}\left[e^{-2\nu k^2(t-t_o)} - e^{-(\kappa+\nu)k^2(t-t_o)}\right] +$$

$$\Lambda_o(k,\mu)e^{-(\kappa+\nu)k^2(t-t_o)} \tag{36}$$

$$\Lambda'(k,\mu,t) = \Lambda_o'(k,\mu)e^{-(\kappa+\nu)k^2(t-t_o)} \tag{37}$$

$$\Theta(k,\mu,t) = \frac{\beta^2\left(1 - \mu^2\right)E_o(k)}{2(\kappa - \nu)^2k^4}\left[e^{-\nu k^2(t-t_o)} - e^{-\kappa k^2(t-t_o)}\right]^2 +$$

$$\left[\Theta_o(k,\mu) + \frac{4\pi\beta\left(1 - \mu^2\right)}{\kappa - \nu}\Lambda_o(k,\mu)\right]e^{-2\kappa k^2(t-t_o)} -$$

$$\frac{4\pi\beta\left(1 - \mu^2\right)}{\kappa - \nu}\Lambda_o(k,\mu)e^{-(\kappa+\nu)k^2(t-t_o)} \tag{38}$$

All variables have been explicitly indicated for emphasis, and $E_o(k) = E(k,t_o)$, $\Lambda_o(k,\mu) = \Lambda(k,\mu,t_o)$, and so forth. From the discussion in the previous section, it may be expected that for $t \geqq t_o$ the errors in these results (as compared with the exact results) are $O\left(R_\lambda\right)$ in the first and $O\left(P_{\lambda\vartheta}\right)$ in the others.

Asymptotic Behavior of Solutions for Large Values of Time

In the preceding solutions there is still a considerable degree of indeterminacy due to the presence of the functions E_o, Θ_o, Λ_o, and Λ_o'. These express the ultimate influence of the past history of the velocity-temperature field during the period of time from the initial instant of generation to the time t_o, when the inertial and pressure effects are significant. For sufficiently large values of $t - t_o$, however, it will be shown that the results given by equations (35) to (38) take on much simpler forms asymptotically, which depend only to a slight extent on the previous history. The asymptotic forms turn out to depend critically on the behavior of the functions E, Θ, Λ, and Λ' for small values of k and thus on the behavior of the corresponding correlation functions for large values of r. A complete analysis of these questions is complicated and lengthy and requires the use of the exact equations of the velocity and temperature fields. Detailed answers are available only for the velocity field alone, in the recent work of Batchelor and Proudman (ref. 12).

The results of reference 12 indicate that for small values of k the behavior of E is in general given by

$$E(k,t) = C(t)k^4 + O\left(k^5 \log_e k\right) \tag{39}$$

On the other hand, the exponential factor in equation (35) becomes very small as $t - t_o \rightarrow \infty$ for all except very small values of k, so that the asymptotic form is obtained by replacing $E_o = E(k,t_o)$ by its leading term when k is small. Thus, the asymptotic form for E as $t - t_o \rightarrow \infty$ is

$$E(k,t) \approx C_o k^4 e^{-2\nu k^2(t-t_o)} \tag{40}$$

where $C_o = C(t_o)$. This is the well-known result for the energy spectrum function E during the final period of decay.

In the present problem of the combined temperature-velocity field, similar results can be obtained for Θ, Λ, and Λ', providing that plausible assumptions are introduced at an appropriate stage in the analysis.

Equations (22) and (24) are first considered and put into more usable forms by the introduction of spherical coordinates (r, α, β), with the aid of the definitions

$$\left.\begin{aligned}
\underset{\sim}{r}\cdot\underset{\sim}{r} &= r^2 \\
\underset{\sim}{r}\cdot\underset{\sim}{\lambda} &= r \cos \alpha = rm \\
\underset{\sim}{k}\cdot\underset{\sim}{\lambda} &= k \cos \gamma = k\mu \\
\underset{\sim}{k}_p\cdot\underset{\sim}{r}_p &= kr\mu m \\
\underset{\sim}{k}_n\cdot\underset{\sim}{r}_n &= kr\sqrt{\left(1 - m^2\right)\left(1 - \mu^2\right)} \cos \beta
\end{aligned}\right\} \tag{41}$$

That is, $\underset{\sim}{k}_p$ and $\underset{\sim}{r}_p$ are the projections of the vectors $\underset{\sim}{k}$ and $\underset{\sim}{r}$ along $\underset{\sim}{\lambda}$, and $\underset{\sim}{k}_n$ and $\underset{\sim}{r}_n$ are their projections in a plane normal to $\underset{\sim}{\lambda}$, so that $\underset{\sim}{k} = \underset{\sim}{k}_p + \underset{\sim}{k}_n$ and $\underset{\sim}{r} = \underset{\sim}{r}_p + \underset{\sim}{r}_n$. With the aid of the formula (ref. 13, p. 87)

$$\int_{-\pi}^{\pi} e^{-iA \cos \beta}\, d\beta = 2\pi J_o(A) \tag{42}$$

where J_o is the Bessel function of the first kind of order zero, the integration with respect to the angle β can be immediately carried out, leading to the results

$$\frac{\Theta(k,\mu)}{2\pi k^2} = (2\pi)^{-2}\int_0^{\infty}\int_{-1}^{1} r^2\theta(r,m) \cos(krm\mu)\, J_o\left[kr\sqrt{\left(1 - m^2\right)\left(1 - \mu^2\right)}\right] dm\, dr \tag{43}$$

$$\Lambda(k,\mu) = \frac{(2\pi)^{-2}}{1 - \mu^2}\int_0^\infty\int_{-1}^1 r^2 L_p(r,m)\ \cos(krm\mu)\ J_o\left[kr\sqrt{\left(1 - m^2\right)\left(1 - \mu^2\right)}\right] dm\ dr \tag{44}$$

$$\Lambda'(k,\mu) = \frac{-i(2\pi)^{-2}}{1 - \mu^2}\int_0^\infty\int_{-1}^1 r^2 L_p'(r,m)\ \sin(krm\mu)\ J_o\left[kr\sqrt{\left(1 - m^2\right)\left(1 - \mu^2\right)}\right] dm\ dr \tag{45}$$

In these relations,

$$\left.\begin{aligned} L_p &= \lambda_i L_i \\ L_p' &= \lambda_i L_i' \end{aligned}\right\} \tag{46}$$

and use has been made of the fact that θ and L_p are even functions of m and L_p' is an odd function of m. These properties are easily derived from relations (10) to (12) and (17). Thus, both Θ and Λ are real and even in μ, and Λ' is purely imaginary and odd in μ.

On a purely formal basis for the moment, the first few derivatives with respect to k of Θ, Λ, and Λ' at $k = 0$ are now determined from relations (43) to (45). In the case of Θ, the results are immediately seen to be

$$\Theta(0,\mu) = \frac{\partial\Theta}{\partial k}(0,\mu) = 0 \tag{47}$$

$$\frac{\partial^2 \Theta}{\partial k^2}(0,\mu) = 2(2\pi)^{-1} \int_0^\infty \int_{-1}^1 r^2 \theta(r,m) \, dm \, dr \tag{48}$$

For the quantities Λ and Λ', the following relation (indicated by Mélèse, ref. 8) is required:

$$L_p = \frac{1 - m^2}{r^2} \frac{\partial}{\partial r} r^3 L - \frac{\partial}{\partial m}\left[m\left(1 - m^2\right)L\right] \tag{49}$$

A similar relation for L_p' follows from equations (17) to (21). With the use of equation (49), and with the assumptions that $r^3 L \to 0$ and $r^3 L' \to 0$ as $r \to \infty$, the following results are obtained:

$$\Lambda(0,\mu) = 0 \tag{50}$$

$$\frac{\partial \Lambda}{\partial k}(0,\mu) = 0 \tag{51}$$

$$\frac{\partial^2 \Lambda}{\partial k^2}(0,\mu) = -\frac{(2\pi)^{-2}}{1 - \mu^2} \int_0^\infty \int_{-1}^1 r^4 L_p(r,m) \left[m^2 \mu^2 + \frac{1}{2}\left(1 - m^2\right)\left(1 - \mu^2\right)\right] dm \, dr \tag{52}$$

$$\Lambda'(0,\mu) = 0 \tag{53}$$

$$\frac{\partial \Lambda'}{\partial k}(0,\mu) = -i \frac{(2\pi)^{-2}\mu}{1 - \mu^2} \int_0^\infty \int_{-1}^1 r^3 m L_p'(r,m) \, dm \, dr \tag{54}$$

(Relations (50) and (53) were obtained previously by Mélèse.) Hence, the behavior of Θ, Λ, and Λ', respectively, for small values of k is given by

$$\Theta(k,\mu,t) = C'(t)k^2 + o(k^2) \tag{55}$$

$$\Lambda(k,\mu,t) = C''(\mu,t)k^2 + o(k^2) \tag{56}$$

$$\Lambda'(k,\mu,t) = C'''(\mu,t)k + o(k) \tag{57}$$

where

$$\left.\begin{aligned} C'(t) &= \frac{1}{2}\left(\frac{\partial^2\Theta}{\partial k^2}\right)_0 \\ C''(\mu,t) &= \frac{1}{2}\left(\frac{\partial^2\Lambda}{\partial k^2}\right)_0 \\ C'''(\mu,t) &= \left(\frac{\partial\Lambda'}{\partial k}\right)_0 \end{aligned}\right\} \tag{58}$$

(See eqs. (48), (52), and (54).)

The formal results derived above will be valid if the integrals involved in equations (47) to (54) actually exist and if the assumptions $r^3L \rightarrow 0$ and $r^3L' \rightarrow 0$ as $r \rightarrow \infty$ are valid. For the purposes of the present investigation, these conditions will be assumed to be satisfied.[1] As mentioned previously, a complete justification would require an elaborate analysis of the behavior of the correlations of temperature and velocity for large values of r, after the manner of the work of Batchelor and Proudman (ref. 12). Although such an investigation has not been carried out in full, a partial study reveals no essentially new difficulties

[1]Sufficient conditions for the existence of the integrals are that $r^5L_p \rightarrow 0$ and $r^4L_p' \rightarrow 0$ as $r \rightarrow \infty$. These are not necessary conditions, however, and it is assuming less just to assume that the integrals exist.

and tends to support the assumptions stated. The latter, moreover, appear quite reasonable from an intuitive point of view.

By the same type of reasoning as that employed in deriving relation (40), it now follows that the asymptotic forms of relations (36) to (38) for large values of $t - t_o$ are obtained by replacing E_o, Λ_o, Λ_o', and Θ_o by their leading terms when k is small, as given by equations (39) and (55) to (57). Hence, for $t - t_o \rightarrow \infty$, relation (36), for example, becomes

$$\Lambda(k,\mu,t) \approx -\frac{\beta C_o}{4\pi(\kappa - \nu)}\left[e^{-2\nu k^2(t-t_o)} - e^{-(\kappa+\nu)k^2(t-t_o)}\right] + C_o''(\mu)k^2 e^{-(\kappa+\nu)k^2(t-t_o)} \tag{59}$$

where $C_o = C(t_o)$ and $C_o''(\mu) = C''(\mu,t_o)$. Analogous results follow for relations (37) and (38). A further study of these new formulas now shows that actually the leading terms are dominant. Consider equation (59), for example, and introduce the nondimensional variable $x = k\sqrt{2\nu(t - t_o)}$ and the reference value

$$\Lambda_m = -\frac{\beta C_o}{4\pi(\kappa - \nu)}\left(e^{-x_1^2} - e^{-\frac{1+\sigma}{2\sigma} x_1^2}\right) \tag{60}$$

where $\sigma = \nu/\kappa$ is the Prandtl number, and x_1 is defined by

$$x_1 = \left(\frac{2\sigma}{1 - \sigma} \log_e \frac{1 + \sigma}{2\sigma}\right)^{1/2} \tag{61}$$

That is, Λ_m is the maximum value of the first term in Λ. Then equation (59) can be written as

$$\frac{\Lambda}{\Lambda_m} = \frac{e^{-x^2} - e^{-\frac{1+\sigma}{2\sigma}x^2}}{e^{-x_1^2} - e^{-\frac{1+\sigma}{2\sigma}x_1^2}} + \frac{C_o''(\mu)}{2\Lambda_m \nu(t - t_o)} x^2 e^{-\frac{1+\sigma}{2\sigma}x^2} \tag{62}$$

Since the second term is $O\left[1/(t - t_o)\right]$, the final result for the asymptotic form of Λ when $t - t_o \rightarrow \infty$ is seen to be

$$\Lambda(k,\mu,t) \approx -\frac{\beta C_o}{4\pi(\kappa - \nu)}\left[e^{-2\nu k^2(t-t_o)} - e^{-(\kappa+\nu)k^2(t-t_o)}\right] \tag{63}$$

Similarly, if Θ_m is defined as the maximum value of the first term in the expression for Θ, it is found that the remaining terms in the expression for Θ/Θ_m are $O(1/(t - t_o))$. Thus, the asymptotic form of Θ when $t - t_o \rightarrow \infty$ is

$$\Theta(k,\mu,t) \approx \frac{\beta^2\left(1 - \mu^2\right)C_o}{2(\kappa - \nu)^2}\left[e^{-\nu k^2(t-t_o)} - e^{-\kappa k^2(t-t_o)}\right]^2 \tag{64}$$

Finally, the quantity Λ' is of interest only in the combination $\Lambda + \Lambda'$, which determines the Fourier transform of the correlation $\vartheta u_i'$ (see eqs. (11) and (12)). A consideration of the nondimensional quantity $(\Lambda + \Lambda')/\Lambda_m$ shows, with the aid of equation (57), that $\Lambda'/\Lambda_m = O\left(1\Big/\sqrt{t - t_o}\right)$. Hence, when $t - t_o \rightarrow \infty$,

$$\Lambda + \Lambda' \approx \Lambda \tag{65}$$

where Λ is given by equation (63).

During the final period of decay, therefore, the asymptotic solutions do not depend on the initial conditions for the temperature field but only

on those for the velocity field, through the parameters C_0 and t_0. In the following sections, various properties of the temperature field will be derived on the basis of the approximate relations (63) to (65).

Some Integral Properties of the Temperature Field

There are several quantities, corresponding to mean values at one point only, that can be expressed entirely in terms of integrals involving the functions Θ and Λ. Before these results for the temperature field are presented, the corresponding results for the velocity field in terms of the energy spectrum function E will be listed for later use.

From equation (40) and the definitions of reference 14, (pp. 47 to 51), it is seen that the mean-square kinetic energy is given by

$$\overline{u^2} = \frac{1}{3}\,\overline{u_i u_i} = \frac{2}{3}\int_0^\infty E(k)\;dk = \frac{\sqrt{2\pi}}{32}\,C_0\left[\nu(t - t_0)\right]^{-5/2} \tag{66}$$

the microscale λ, by

$$\frac{1}{\lambda^2} = \frac{2}{15\overline{u^2}}\int_0^\infty k^2 E(k)\;dk = \frac{1}{4}\left[\nu(t - t_0)\right]^{-1} \tag{67}$$

and the integral scale L, by

$$L = \frac{\pi}{2\overline{u^2}}\int_0^\infty k^{-1}E(k)\;dk = \sqrt{\frac{\pi}{2}}\,\lambda \tag{68}$$

Thus, in all further results it will be possible to express the quantities C_0 and $t - t_0$ in terms of $\left(\overline{u^2}\right)^{1/2}$ and λ, which are of more direct physical significance. The resulting formulas will then contain only quantities that are directly observable at any instant during the final period of decay, in addition to the given constant parameters of the problem (i.e., ν, κ, and β).

Perhaps the quantity of greatest physical interest predicted by the present theory is the level of mean-square temperature fluctuations, which follows from equations (23), (64), (66), and (67) in the forms

$$\overline{\vartheta^2} = \frac{\sqrt{\pi}\beta^2\sigma^2}{3\sqrt{2}\nu^2(1 - \sigma)^2} C_0\left[\nu(t - t_0)\right]^{-1/2}\left[1 - 2\left(\frac{2\sigma}{1 + \sigma}\right)^{1/2} + \sigma^{1/2}\right]$$

$$= \frac{1}{3} \frac{\sigma^2}{(1 - \sigma)^2} R_\lambda{}^2\beta^2\lambda^2\left[1 - 2\left(\frac{2\sigma}{1 + \sigma}\right)^{1/2} + \sigma^{1/2}\right] \tag{69}$$

where $R_\lambda = \left(\overline{u^2}\right)^{1/2} \lambda/\nu$. The first form indicates that $\overline{\vartheta^2}$ decays more slowly than $\overline{u^2}$, just as Corrsin found in the simpler problem with constant mean temperature (ref. 9).

Another quantity of some interest involves the mean-square gradient of ϑ. Thus, if

$$G_i = \frac{\partial\vartheta}{\partial x_i} \tag{70}$$

and if the components of $\underset{\sim}{G}$ parallel $\left(G_p = \lambda_i G_i\right)$ and perpendicular to $\underset{\sim}{\lambda}$ are considered and axisymmetry noted, there results

$$\overline{\underset{\sim}{G}^2} = \overline{G_p{}^2} + 2\overline{G_n{}^2} = \int_0^\infty\int_{-1}^1 k^2\Theta(k,\mu)\, d\mu\, dk \tag{71}$$

The quantities $\overline{G_p{}^2}$ and $\overline{G_n{}^2}$ are, respectively, the mean-square values of the component of G parallel to $\underset{\sim}{\lambda}$ and the component in any direction perpendicular to $\underset{\sim}{\lambda}$ and are given by

$$\overline{G_p{}^2} = \int_0^{\infty}\int_{-1}^{1} k^2\mu^2\Theta(k,\mu)\ d\mu\ dk$$

$$= \frac{\sqrt{\pi}\beta^2\sigma^2}{60\sqrt{2}\nu^2(1 - \sigma)^2} C_o\left[\nu(t - t_o)\right]^{-3/2}\left[1 - 2\left(\frac{2\sigma}{1 + \sigma}\right)^{3/2} + \sigma^{3/2}\right]$$

$$= \frac{1}{15}\frac{\sigma^2}{(1 - \sigma)^2} R_{\lambda}{}^2\beta^2\left[1 - 2\left(\frac{2\sigma}{1 + \sigma}\right)^{3/2} + \sigma^{3/2}\right] \tag{72}$$

and

$$\overline{G_n{}^2} = \int_0^{\infty}\int_{-1}^{1} k^2\left(1 - \mu^2\right)\Theta(k,\mu)\ d\mu\ dk = 2\overline{G_p{}^2} \tag{73}$$

From these results one can, of course, obtain the mean-square gradient of ϑ in an arbitrary direction with respect to $\underset{\sim}{\lambda}$. Relation (73) expresses the physical fact that the mean square of the molecular heat transfer fluctuations is greater "across the mean gradient" than it is "down the mean gradient," in a ratio that is independent of β. This conclusion is rather surprising, in view of the fact that the mean heat transfer (both molecular and turbulent) is entirely in the direction of the mean temperature gradient.

Consider now the cross correlation $L_i(\underset{\sim}{r},\underset{\sim}{\lambda})$. It is found, for example, that

$$\overline{\vartheta u_p} = \lambda_i L_i(0,\underset{\sim}{\lambda}) = 2\pi\int_0^{\infty}\int_{-1}^{1}\left(1 - \mu^2\right)\Lambda(k,\mu)\ d\mu\ dk \tag{74}$$

and

$$\overline{\vartheta u_n} \equiv 0 \tag{75}$$

corresponding to the fact that the heat-transfer vector $L_i(0,\underset{\sim}{\lambda})$ is different from zero only in the direction of $\underset{\sim}{\lambda}$. From equations (63) and (74) follows the explicit relation

$$\begin{aligned}\overline{\vartheta u_p} &= -\frac{\sqrt{2\pi}\beta\sigma}{24\nu(1-\sigma)} C_o\left[\nu(t-t_o)\right]^{-3/2}\left[1-\left(\frac{2\sigma}{1+\sigma}\right)^{3/2}\right] \\ &= -\frac{1}{3}\frac{\sigma}{1-\sigma}R_\lambda\left(\overline{u^2}\right)^{1/2}\beta\lambda\left[1-\left(\frac{2\sigma}{1+\sigma}\right)^{3/2}\right]\end{aligned} \tag{76}$$

The heat-transfer correlation coefficient

$$R_h = \frac{\overline{\vartheta u_p}}{\left(\overline{\vartheta^2}\ \overline{u_p{}^2}\right)^{1/2}} \tag{77}$$

introduced by Corrsin (ref. 1) now becomes[2]

$$R_h = -\frac{1}{\sqrt{3}}\left|1-\left(\frac{2\sigma}{1+\sigma}\right)^{3/2}\right|\left|\left[1-2\left(\frac{2\sigma}{1+\sigma}\right)^{1/2}+\sigma^{1/2}\right]^{-1/2}\right. \tag{78}$$

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[2]For isotropic turbulence, $\overline{u_p{}^2} = \overline{u^2}$. Also, note that β is by definition a positive constant.

A quantity of particular physical significance is the ratio of mean turbulent heat transfer to mean molecular heat transfer given by

$$\frac{-\overline{\vartheta u_p}}{\kappa\beta} = \frac{1}{3} R_\lambda^{\,2} \frac{\sigma^2}{1 - \sigma}\left[1 - \left(\frac{2\sigma}{1 + \sigma}\right)^{3/2}\right] \tag{79}$$

The behavior of this ratio is evidently just what would be expected physically. For a given turbulent motion (i.e., a given value of R_λ), the ratio decreases with decreasing values of σ (i.e., increasing thermal conductivity of the fluid). On the other hand, for given fluid properties (i.e., a given value of σ), it increases with increasing Reynolds number R_λ of the turbulent motion.

For the correlation $\overline{G_i\omega_j}$, where $\omega_j = \epsilon_{jkl}\left(\partial u_l/\partial x_k\right)$, some rather involved calculations lead to the relatively simple result

$$\overline{G_i\omega_j} = -\frac{1}{2}\frac{\sigma}{1 - \sigma}\left[1 - \left(\frac{2\sigma}{1 + \sigma}\right)^{5/2}\right] R_\lambda \frac{\beta\left(\overline{u^2}\right)^{1/2}}{\lambda}\epsilon_{ijk}\lambda_k \tag{80}$$

which is different from zero if and only if $\underset{\sim}{G}$ and $\underset{\sim}{\omega}$ are perpendicular to each other as well as to $\underset{\sim}{\lambda}$. After the general tensor relation for $\overline{G_i\omega_j}$ in terms of Λ is obtained, the above relation is derived most readily by taking rectangular coordinates $\left(x_1,\ x_2,\ \text{and}\ x_3\right)$ with x_1 along $\underset{\sim}{\lambda}$. It can then be shown that $\overline{G_1\omega_1} = \overline{G_2\omega_2} = \overline{G_3\omega_3} = 0$ and, with the further use of spherical coordinates $(k,\ \gamma,\ \text{and}\ \beta)$ as described by the defining relations (41), that

$$\overline{G_2\omega_3} = -\overline{G_3\omega_2} = \int k_2{}^2\Lambda\, d\underset{\sim}{k} = \frac{4\pi}{3}\int_0^\infty k^4\Lambda\, dk \tag{81}$$

From these results and equation (63), equation (80) follows.

Another quantity of some physical interest is the microscale or "dissipation" length parameter λ_ϑ for the temperature field, first introduced by Corrsin (ref. 9) and defined by

$$\overline{\underset{\sim}{G}^2} = -\left[\frac{\partial^2\theta(\underset{\sim}{r},\underset{\sim}{\lambda})}{\partial r_i\ \partial r_i}\right]_{\underset{\sim}{r}=0} = 6\ \frac{\overline{\vartheta^2}}{\lambda_\vartheta{}^2} \tag{82}$$

In the present case, equation (82) reduces to

$$\frac{\lambda_\vartheta{}^2}{\lambda^2} = 6\left[1 - 2\left(\frac{2\sigma}{1 + \sigma}\right)^{1/2} + \sigma^{1/2}\right]\left[1 - 2\left(\frac{2\sigma}{1 + \sigma}\right)^{3/2} + \sigma^{3/2}\right]^{-1} \tag{83}$$

It should be noted that the geometrical interpretation of λ_ϑ is now not so simple as it was in the isotropic case (ref. 9). In fact, with use of the relation $m = r_k\lambda_k/r$, it is found that

$$\frac{\partial^2\theta(\underset{\sim}{r},\underset{\sim}{\lambda})}{\partial r_i\ \partial r_i} = \frac{\partial^2\theta(r,m)}{\partial r^2} + \frac{2}{r}\ \frac{\partial\theta(r,m)}{\partial r} + \frac{\partial}{\partial m}\left[\frac{1 - m^2}{r^2}\ \frac{\partial\theta(r,m)}{\partial m}\right] \tag{84}$$

from which follows

$$\left[\frac{\partial^2\theta(\underset{\sim}{r},\underset{\sim}{\lambda})}{\partial r_i\ \partial r_i}\right]_{\underset{\sim}{r}=0} = 3\left[\frac{\partial^2\theta(r,m)}{\partial r^2}\right]_{r=0} + \frac{1}{2}\ \frac{\partial}{\partial m}\left\{\left(1 - m^2\right)\frac{\partial}{\partial m}\left[\frac{\partial^2\theta(r,m)}{\partial r^2}\right]_{r=0}\right\} \tag{85}$$

However, averaging this equation with respect to m leads to the simple result

$$\left[\frac{\partial^2\theta(\underset{\sim}{r},\underset{\sim}{\lambda})}{\partial r_i\ \partial r_i}\right]_{\underset{\sim}{r}=0} = 3\int_{-1}^{1}\left[\frac{\partial^2\theta(r,m)}{\partial r^2}\right]_{r=0} dm \tag{86}$$

Thus, in an average sense, λ_ϑ is still a representative length scale for the variation of $\theta(r,m)$ in the vicinity of $r = 0$.

Another characteristic length for the temperature field is the thermal integral scale

$$L_\vartheta = \frac{1}{\overline{\vartheta^2}} \int_0^\infty \theta(r,m)\, dr \tag{87}$$

also introduced by Corrsin (ref. 9). This quantity is expected to give a measure of the distance over which the temperature fluctuations are effectively correlated, as the integral scale L does for the velocity fluctuations. When θ is expressed as the inverse of the transform relation (22) and the spherical coordinates (k, γ, and β) in the $\underset{\sim}{k}$ space described previously are used, equation (87) takes the form

$$L_\vartheta = \frac{2}{\overline{\vartheta^2}} \int_0^\infty \int_0^\infty \int_0^1 \Theta(k,\mu) \cos(krm\mu)\, J_0\left[kr\sqrt{\left(1 - m^2\right)\left(1 - \mu^2\right)}\right] d\mu\, dk\, dr \tag{88}$$

The integration with respect to r can be immediately carried out (see ref. 15, p. 43), leading to

$$L_\vartheta = \frac{2}{\overline{\vartheta^2}} \int_0^\infty \int_0^{\sqrt{1-m^2}} \frac{\Theta(k,\mu)}{k}\left(1 - m^2 - \mu^2\right)^{-1/2} d\mu\, dk \tag{89}$$

When the expression (64) for Θ is introduced, the integration with respect to μ is straightforward. The final integration with respect to k is readily performed by integrating from k_o to ∞ and then taking the limit as $k_o \to 0$, with the aid of the formula

$$\int_{x_o}^{\infty} \frac{e^{-Ax^2}}{x}\,dx = -e^{-Ax_o^2}\log_e x_o + 2\int_{x_o\sqrt{A}}^{\infty} xe^{-x^2}\log_e x\,dx -$$

$$2\log_e \sqrt{A}\int_{x_o\sqrt{A}}^{\infty} xe^{-x^2}\,dx \tag{90}$$

where $x = k\sqrt{2\nu(t - t_o)}$. The end result, after elimination of $\overline{\vartheta^2}$ by equation (69), is

$$L_\vartheta = \frac{3\sqrt{2\pi}}{16}\lambda\left(1 + m^2\right)\left[1 - 2\left(\frac{2\sigma}{1 + \sigma}\right)^{1/2} + \sigma^{1/2}\right]^{-1}\log_e \frac{(1 + \sigma)^2}{4\sigma} \tag{91}$$

An examination of the preceding results indicates that in this problem, for a given field of turbulence, many of the thermal characteristics are strongly dependent on the Prandtl number. It is instructive, therefore, to present a summary of the formulas derived above in the special cases: $\sigma \gg 1$ (corresponding to several liquids, such as water and lubricating oils), $\sigma = 1$ (approximately representative of gases), and $\sigma \ll 1$ (corresponding to certain liquid metals).

For $\sigma \gg 1$, then, the following approximate relations result:

$$\left.\begin{aligned}
&\overline{\vartheta^2} \cong \frac{1}{3}\sigma^{1/2}R_\lambda{}^2\beta^2\lambda^2, && \overline{G_p{}^2} \cong \frac{1}{15}\sigma^{3/2}R_\lambda{}^2\beta^2 \\
&\overline{\vartheta u_p} \cong -\frac{2\sqrt{2} - 1}{3}R_\lambda\left(\overline{u^2}\right)^{1/2}\beta\lambda, && R_h \cong -\frac{2\sqrt{2} - 1}{\sqrt{3}}\sigma^{-1/4} \\
&-\frac{\overline{\vartheta u_p}}{\kappa\beta} \cong \frac{2\sqrt{2} - 1}{3}R_\lambda{}^2\sigma, && \overline{G_i\omega_j} \cong -\frac{4\sqrt{2} - 1}{2}R_\lambda\frac{\beta\left(\overline{u^2}\right)^{1/2}}{\lambda}\epsilon_{ijk}\lambda_k \\
&\frac{\lambda_\vartheta}{\lambda} \cong \left(\frac{6}{\sigma}\right)^{1/2}, && \frac{L_\vartheta}{\lambda} \cong \frac{3\sqrt{2\pi}}{16}\left(1 + m^2\right)\frac{\log_e \sigma}{\sigma^{1/2}}
\end{aligned}\right\} \tag{92}$$

Perhaps the most noteworthy feature here is that λ_ϑ and L_ϑ are both much smaller than λ, indicating that the temperature correlation curve is much more strongly curved near the origin than the velocity correlation curve and falls off to zero much more rapidly. It should be emphasized, of course, that this case places rather severe limitations on R_λ. Since the present theory is restricted to small values of $P_{\lambda_\vartheta} = \left(\sigma\lambda_\vartheta/\lambda\right)R_\lambda \cong \sqrt{6\sigma}R_\lambda$, the Reynolds number R_λ has to be extremely small if both the conditions $\sigma \gg 1$ and $\sqrt{6\sigma}R_\lambda \ll 1$ are to be satisfied.

For $\sigma = 1$, the formulas become

$$\left.\begin{aligned}
&\overline{\vartheta^2} = \frac{1}{16} R_\lambda{}^2\beta^2\lambda^2, && \overline{G_p{}^2} = \frac{1}{16} R_\lambda{}^2\beta^2 \\
&\overline{\vartheta u_p} = -\frac{1}{4} R_\lambda\left(\overline{u^2}\right)^{1/2}\beta\lambda, && R_h = -1 \\
&\frac{-\overline{\vartheta u_p}}{\kappa\beta} = \frac{1}{4} R_\lambda{}^2, && \overline{G_i\omega_j} = -\frac{5}{8} R_\lambda \frac{\beta\left(\overline{u^2}\right)^{1/2}}{\lambda} \epsilon_{ijk}\lambda_k \\
&\frac{\lambda_\vartheta}{\lambda} = \frac{6}{5}, && \frac{L_\vartheta}{\lambda} = \frac{\sqrt{2\pi}}{4}\left(1 + m^2\right)
\end{aligned}\right\} \tag{93}$$

In this case, the characteristic lengths for the temperature field are of the same order of magnitude as those for the velocity field. The result for R_h is very interesting, since it implies that the fluctuations ϑ and $-u_p = -u_i\lambda_i$ at a point are perfectly correlated. (Note that $-u_p$ is the velocity component in the direction of <u>decreasing</u> mean temperature.)

For $\sigma \ll 1$, the following approximate formulas are obtained:

$$\left.\begin{aligned}
&\overline{\vartheta^2} \cong \frac{1}{3}\,\sigma^2 R_\lambda{}^2 \beta^2 \lambda^2, &&\overline{G_p{}^2} \cong \frac{1}{15}\,\sigma^2 R_\lambda{}^2 \beta^2 \\
&\overline{\vartheta u_p} \cong -\frac{1}{3}\,\sigma R_\lambda \left(\overline{u^2}\right)^{1/2} \beta\lambda, &&R_h \cong -\frac{1}{\sqrt{3}} \\
&\frac{-\overline{\vartheta u_p}}{\kappa\beta} \cong \frac{1}{3}\, R_\lambda{}^2 \sigma^2, &&\overline{G_i \omega_j} \cong -\frac{1}{2}\,\sigma R_\lambda\, \frac{\beta\left(\overline{u^2}\right)^{1/2}}{\lambda}\, \epsilon_{ijk}\lambda_k \\
&\frac{\lambda_\vartheta}{\lambda} \cong 6, &&\frac{L_\vartheta}{\lambda} \cong \frac{3\sqrt{2\pi}}{16}\left(1 + m^2\right) \log_e \frac{1}{\sigma}
\end{aligned}\right\} \qquad (94)$$

Although λ_ϑ is of the same order of magnitude as λ in this case, L_ϑ is much larger, indicating that the temperature correlation function falls off rather slowly for large separation distances. In fact, the decrease of $\theta(r,m)$ for large values of r is so slow in the limiting case $\sigma \to 0$ that the integral in equation (87) becomes divergent and L_ϑ becomes infinite. Thus, in this limiting situation the quantity L_ϑ loses its significance as a representative length for the temperature field. The length λ_ϑ remains significant, however, as a measure of the radius of curvature of the temperature correlation function near the origin. In the next section, a more detailed analysis of the behavior of the correlation functions for the temperature field will be presented.

A final observation of considerable interest is that, if the arbitrary assumption is made initially that the turbulence is statistically stationary[3], then the final results agree with relations (94) when σ is small. The above results for λ_ϑ and R_h were obtained by Mélèse (ref. 8), and most of the others were obtained in a preliminary version of the

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[3]This implies that any mean value involving fluctuating quantities at different times is dependent only on the time differences and not on the individual times. Thus, mean values involving quantities at the same time, such as those in the present investigation, are independent of the time. The term "nondecaying" is also used in this sense.

present work, all on the basis of this hypothesis. Although stationary <u>isotropic</u> turbulence is impossible, strictly speaking, its assumption evidently leads to a formulation of the problem that is a valid approximation when the Prandtl number is small.

A clearer picture of the situation can be obtained as follows. First, it is noted that the assumption of stationary turbulence implies that the temperature field is also stationary, so that the time-derivative terms in equations (13) to (15) are absent and $L_i' \equiv 0$. On the other hand, if the assumption is not made, it is found that $\partial\overline{Q}/\partial t = O(\overline{Q}/(t - t_o))$ for any correlation $\overline{Q}$ involving the temperature, so that in each of equations (13) and (14) the time-derivative term is $O\left(\sigma\lambda_\vartheta^2/\lambda^2\right)$ relative to the heat-conduction term and is thus negligible when σ is small. Moreover, it is also found that $L_i + L_i' \approx L_i$ as $t - t_o \rightarrow \infty$. Hence, as far as the solution of the problem during the final period of decay is concerned, the basic equations obtained with the assumption of stationarity are valid approximations for small values of σ, providing the correlation function $f(r/\lambda)$ describing the turbulence is properly specified.

Physically, when the Prandtl number is small, the effects of time variation are negligible, and the dissipation through conduction is balanced primarily by the convective heat transfer associated with the mean temperature gradient. The dominating influence of the mean temperature gradient in this situation appears to be the reason for the basic difference between relation (94) for $\lambda_\vartheta/\lambda$ and the result obtained by Corrsin (ref. 9) in the problem with zero mean temperature gradient, namely, that over a wide range of conditions $\lambda_\vartheta/\lambda = O\left(\sigma^{-1/2}\right)$ for all Prandtl numbers. In the latter problem, the time variation is <u>always</u> a dominant factor.

Some Detailed Properties of the Temperature Field

The results of the preceding section were concerned with mean values at a single point and yielded only limited information about the various correlations as functions of the spatial separation. The characteristics of the temperature field associated with the two-point mean values are now considered in more detail.

The scalar Fourier transform relation between $\theta(r,m)$ and $\Theta(k,\mu)$ which follows from equation (22) is

$$\theta(r,m) = 2\int_0^\infty \int_0^1 \Theta(k,\mu)\ \cos(krm\mu)\ J_o\left[kr\sqrt{\left(1 - m^2\right)\left(1 - \mu^2\right)}\right]\, d\mu\ dk \tag{95}$$

If a correlation coefficient for the temperature field is now defined in the usual way as

$$M(r,m) = \frac{\overline{\vartheta\vartheta'}}{\overline{\vartheta^2}} = \frac{\theta(r,m)}{\theta(0,m)} \tag{96}$$

then using equations (64) and (95) gives

$$M(y,m) = \frac{\int_0^\infty \int_0^1 \left(1 - \mu^2\right)\left(e^{-x^2/2} - e^{-x^2/2\sigma}\right)^2 \cos(xy\mu m)\ J_o\left[xy\sqrt{\left(1 - m^2\right)\left(1 - \mu^2\right)}\right] d\mu\ dx}{\int_0^\infty \int_0^1 \left(1 - \mu^2\right)\left(e^{-x^2/2} - e^{-x^2/2\sigma}\right)^2 d\mu\ dx} \tag{97}$$

where $y = r\Big/\sqrt{2\nu(t - t_o)} = \sqrt{2}r/\lambda$. The integrals occurring in this expression can be evaluated quite easily (e.g., see ref. 15, p. 57, formula (52)) and the result written in the form

$$M(y,m) = M_1(y) + m^2 M_2(y) \tag{98}$$

where

$$M_1(y) = \frac{M_{11}(y) - 2\sqrt{\frac{2\sigma}{1 + \sigma}}\, M_{11}\left(y\sqrt{\frac{2\sigma}{1 + \sigma}}\right) + \sqrt{\sigma}M_{11}\left(y\sqrt{\sigma}\right)}{1 - 2\sqrt{\frac{2\sigma}{1 + \sigma}} + \sqrt{\sigma}} \tag{99}$$

$$M_2(y) = \frac{M_{21}(y) - 2\sqrt{\frac{2\sigma}{1 + \sigma}}\, M_{21}\left(y\sqrt{\frac{2\sigma}{1 + \sigma}}\right) + \sqrt{\sigma} M_{21}\left(y\sqrt{\sigma}\right)}{1 - 2\sqrt{\frac{2\sigma}{1 + \sigma}} + \sqrt{\sigma}} \tag{100}$$

$$M_{11}(y) = \frac{3}{4}\frac{\sqrt{\pi}}{y}\left\{\operatorname{erf}(y/2) + \frac{1}{y}\left[\frac{\operatorname{erf}(y/2)}{y/2} - \frac{2}{\sqrt{\pi}} e^{-y^2/4}\right]\right\} \tag{101}$$

and

$$M_{21}(y) = \frac{3}{4}\frac{\sqrt{\pi}}{y}\left\{\operatorname{erf}(y/2) - \frac{3}{y}\left[\frac{\operatorname{erf}(y/2)}{y/2} - \frac{2}{\sqrt{\pi}} e^{-y^2/4}\right]\right\} \tag{102}$$

The symbol $\operatorname{erf}(\xi)$ denotes the error function $\frac{2}{\sqrt{\pi}}\int_0^{\xi} e^{-\xi^2}\, d\xi$.

For small values of y, $M(y,m)$ has the typical parabolic behavior

$$M(y,m) = 1 - \frac{1}{10}\left(1 - \frac{m^2}{2}\right)\frac{1 - 2\left(\frac{2\sigma}{1 + \sigma}\right)^{3/2} + \sigma^{3/2}}{1 - 2\left(\frac{2\sigma}{1 + \sigma}\right)^{1/2} + \sigma^{1/2}}\, y^2 + O\left[y^4\left(1 + \sigma^2\right)\right] \tag{103}$$

while for large values of y it decreases in the manner

$$M(y,m) = -\frac{3(1 - m^2)}{y^2}\,\frac{e^{-y^2/4} - 2\left(\frac{2\sigma}{1 + \sigma}\right)^{-1/2} e^{-\sigma y^2/2(1+\sigma)} + \sigma^{-1/2} e^{-\sigma y^2/4}}{1 - 2\left(\frac{2\sigma}{1 + \sigma}\right)^{1/2} + \sigma^{1/2}}\left[1 + O\left(\frac{1 + \sigma}{\sigma y^2}\right)\right] \tag{104}$$

It is noteworthy that, for $\sigma \neq 0$, $M(y,m)$ always becomes negative for sufficiently large values of y.

The limiting form of $M(y,m)$ for zero Prandtl number is of particular interest, in view of the remarks made in the last section. For $\sigma \to 0$, in fact,

$$M(y,m) = M_{11}(y) + m^2 M_{21}(y) + O\left(\sigma^{1/2}\right) \tag{105}$$

For small values of y, the limit function $M_o(y,m) = M_{11}(y) + m^2 M_{21}(y)$ has the parabolic behavior

$$M_o(y,m) = 1 - \frac{1}{10}\left(1 - \frac{m^2}{2}\right)y^2 + O\left(y^4\right) \tag{106}$$

but for large values of y it decreases remarkably slowly, in the manner

$$M_o(y) = \frac{3\sqrt{\pi}}{4} \frac{1 + m^2}{y} + O\left(\frac{1}{y^3}\right) \tag{107}$$

The difference between the limit of relation (104) as $\sigma \to 0$ and relation (107) is due to the fact that the ratio $(M_o - M)/M_o$ does not remain <u>uniformly</u> small for all values of y. For a very small but finite value of σ, the function $M(y,m)$ follows the limit function $M_o(y,m)$ very closely as y increases, but it begins to deviate appreciably when y becomes very large. In fact, it can be shown from equations (104) and (107) that $M_o - M$ is of the order of magnitude of M_o when y is of the order of $\sigma^{-1/2}$. For $\sigma^{1/2}y \gg 1$, $|M|$ decreases to zero much more rapidly than M_o. The decrease of $|M|$ is, in fact, rapid enough to insure the convergence of the integral of $\underline{M}$ (see eq. (87)). On the other hand, equation (107) shows that the integral of the limit function M_o diverges logarithmically. These remarks help to explain the behavior of the thermal integral scale L_ϑ described previously.

The case $\sigma = 1$ is of considerable physical importance, since it corresponds approximately to gases, most of which have Prandtl numbers

slightly less than unity. The result for $M(y,m)$ in this case, which is easily obtained by taking the limit $\sigma \to 1$ in equations (98) to (100), has the especially simple form

$$M(y,m) = e^{-y^2/4}\left(1 - \frac{y^2}{4}\right) + \frac{m^2}{4} y^2 e^{-y^2/4} \tag{108}$$

The functions $M_1(y)$ and $M_2(y)$ have been evaluated numerically for the cases $\sigma = 0, 0.01, 0.1, 0.71, 1, 10$, and 100, respectively. The results for all other cases can be readily obtained from those for $\sigma = 0$ in table 1[4], since the functions $M_1(y)$ and $M_2(y)$ for this case are the functions $M_{11}(y)$ and $M_{21}(y)$ occurring in the general formulas (99) to (102). The numerical values for the case $\sigma = 0.71$, corresponding to air under standard laboratory conditions, are presented in table 3, since they may be of some general interest.

The resultant correlations $M(y,m)$ for $m = 0$ and 1 (corresponding to $\alpha = 90^\circ$ and 0°, respectively) are shown in figures 1(a) to 1(c), with the velocity correlation $f(y) = e^{-y^2/4}$ included in each case for comparison. The correlations for all other values of m lie between those shown. The curves for $\sigma = 0.01$ and 0.1 are included in figure 1 along with those for $\sigma = 0$ in order to illustrate the approach to the limit discussed previously. In contrast to the slow rate of decrease of $M(y,m)$ for small values of σ, the rapid rate of decrease for large values of σ is to be noted. This behavior, it may be recalled, was indicated by the previous results for the thermal integral scale. The physical significance is just that for small values of the Prandtl number, the temperature fluctuations remain significantly correlated over much greater distances than the velocity fluctuations, while for large values the opposite is true.

Consider now the correlation $L_i(\underset{\sim}{r},\underset{\sim}{\lambda})$. It is observed that if

$$\overline{\vartheta u_p'} = \lambda_i L_i(\underset{\sim}{r},\underset{\sim}{\lambda}) \tag{109}$$

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[4]There is a loss of accuracy near $\sigma = 1$ when eqs. (99) and (100) are used directly, with functional values interpolated from table 1.

(see fig. 2(a)), then the Fourier transform (eq. (24)) yields

$$\overline{\vartheta u_p{}'} = 4\pi \int_0^\infty \int_0^1 \left(1 - \mu^2\right) k^2 \Lambda(k,\mu) \cos(krm\mu) \, J_o\left[kr\sqrt{\left(1 - m^2\right)\left(1 - \mu^2\right)}\right] d\mu \, dk \tag{110}$$

Thus, letting

$$N(r,m) = \frac{\overline{\vartheta u_p{}'}}{\overline{\vartheta u_p}} \tag{111}$$

and substituting for $\Lambda(k,\mu)$ from equation (63) give

$$N(y,m) = \frac{\int_0^\infty \int_0^1 \left(1 - \mu^2\right) x^2 \left(e^{-x^2} - e^{-\frac{1+\sigma}{2\sigma}x^2}\right) \cos(xy\mu m) \, J_o\left[xy\sqrt{\left(1 - m^2\right)\left(1 - \mu^2\right)}\right] d\mu \, dx}{\int_0^\infty \int_0^1 \left(1 - \mu^2\right) x^2 \left(e^{-x^2} - e^{-\frac{1+\sigma}{2\sigma}x^2}\right) d\mu \, dx} \tag{112}$$

where $x = k\sqrt{2\nu(t - t_o)}$ and $y = r\Big/\sqrt{2\nu(t - t_o)}$. This can be evaluated to give

$$N(y,m) = N_1(y) + m^2 N_2(y) \tag{113}$$

where

$$N_1(y) = \frac{N_{11}(y) - \left(\frac{2\sigma}{1 + \sigma}\right)^{3/2} N_{11}\left(y\sqrt{\frac{2\sigma}{1 + \sigma}}\right)}{1 - \left(\frac{2\sigma}{1 + \sigma}\right)^{3/2}} \tag{114}$$

$$N_2(y) = \frac{N_{21}(y) - \left(\frac{2\sigma}{1 + \sigma}\right)^{3/2} N_{21}\left(y\sqrt{\frac{2\sigma}{1 + \sigma}}\right)}{1 - \left(\frac{2\sigma}{1 + \sigma}\right)^{3/2}} \tag{115}$$

$$N_{11}(y) = \frac{3}{2}\left\{e^{-y^2/4} - \frac{\sqrt{\pi}}{y^2}\left[\frac{\operatorname{erf}(y/2)}{y/2} - \frac{2}{\sqrt{\pi}} e^{-y^2/4}\right]\right\} \tag{116}$$

and

$$N_{21}(y) = -\frac{3}{2}\left\{e^{-y^2/4} - \frac{3\sqrt{\pi}}{y^2}\left[\frac{\operatorname{erf}(y/2)}{y/2} - \frac{2}{\sqrt{\pi}} e^{-y^2/4}\right]\right\} \tag{117}$$

For small values of y, $N(y,m)$ also has the familiar parabolic behavior

$$N(y,m) = 1 - \frac{3}{10}\left(1 - \frac{m^2}{2}\right)\frac{1 - \left(\frac{2\sigma}{1 + \sigma}\right)^{5/2}}{1 - \left(\frac{2\sigma}{1 + \sigma}\right)^{3/2}} y^2 + O\left[y^4\left(1 + \sigma^2\right)\right] \tag{118}$$

while for large values of y it decreases in the manner

$$N(y,m) = \frac{3(1 - m^2)}{2} \frac{e^{-y^2/4} - \left(\frac{2\sigma}{1 + \sigma}\right)^{3/2} e^{-\sigma y^2/2(1+\sigma)}}{1 - \left(\frac{2\sigma}{1 + \sigma}\right)^{3/2}}\left[1 + O\left(\frac{1 + \sigma}{\sigma y^2}\right)\right] \tag{119}$$

Just as in the case of $M(y,m)$, for $\sigma \neq 0$, the function $N(y,m)$ always becomes negative for sufficiently large values of y.

The limiting form of $N(y,m)$ for small Prandtl numbers is also of interest. For $\sigma \to 0$ the following relation is obtained:

$$N(y,m) = N_{11}(y) + m^2 N_{21}(y) + O\left(\sigma^{3/2}\right) \tag{120}$$

For small values of y, the limit function $N_o(y,m) = N_{11}(y) + m^2 N_{21}(y)$ has the behavior

$$N_o(y,m) = 1 - \frac{3}{10}\left(1 - \frac{m^2}{2}\right)y^2 + O(y^4) \tag{121}$$

For large values of y and for $m \neq 1/\sqrt{3}$ it decreases somewhat faster than $M_o(y,m)$, in the manner

$$N_o(y,m) = 3\sqrt{\pi}\left(3m^2 - 1\right)\frac{1}{y^3} + O\left(e^{-y^2/4}\right) \tag{122}$$

The case $m = 1/\sqrt{3}$, for which

$$N_o\left(y,1/\sqrt{3}\right) = e^{-y^2/4} \tag{123}$$

divides the functions $N_o(y,m)$ into those that are positive everywhere and those that become negative (see fig. 3(a)).[5] The differences between the behavior of the limit function $N_o(y,m)$ for large values of y and that of the actual function $N(y,m)$, for any small but finite value of σ, are to noted. The reasons for these differences are very much like those described for the function $M(y,m)$.

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[5]It might be noted that there is no such dividing correlation when $\sigma \neq 0$, since then the function $N(y,m)$ always becomes negative for large values of y.

In contrast to $M(y,m)$, the correlation $N(y,m)$ has a limiting form for large Prandtl numbers. Thus, when $\sigma \to \infty$,

$$N(y,m) = \left(2\sqrt{2} - 1\right)^{-1}\left[2\sqrt{2}N_{11}\left(y\sqrt{2}\right) - N_{11}(y)\right] + O\left(\frac{1}{\sigma}\right) \tag{124}$$

The limit function in this case is obviously not radically different in shape and asymptotic behavior from the function $N(y,m)$ for a value of σ of order unity.

For $\sigma = 1$, a simple result is again obtained, in fact, exactly the same functional form for $N(y,m)$ as for $M(y,m)$, namely,

$$N(y,m) = e^{-y^2/4}\left(1 - \frac{y^2}{4}\right) + \frac{m^2}{4}\, y^2 e^{-y^2/4} \tag{125}$$

The functions $N_1(y)$ and $N_2(y)$ have been computed for the cases $\sigma = 0$, 0.1, 0.71, 1, 10, and ∞. The numerical values for the cases of most interest, that is, $\sigma = 0$ and $\sigma = 0.71$, are tabulated in tables 2 and 3, respectively. The resultant correlations for $m = 0$ and 1 (corresponding to $\alpha = 90^\circ$ and 0°) are shown in figures 3(a) to 3(c). The velocity correlation $f(y) = e^{-y^2/4}$ is also included in each figure. For $\sigma = 0$, $f(y)$ is the same as $N\left(y,1/\sqrt{3}\right)$ (see eq. (123)).

From these results for the correlation $\overline{\vartheta u_p{}'}$ it is possible to derive the corresponding results for the defining scalars of $L_i\left(\underset{\sim}{r},\underset{\sim}{\lambda}\right)$ and hence to obtain the correlation of ϑ with the velocity in an arbitrary direction. Thus, from equations (17), (19), and (20),

$$\lambda_i L_i\left(\underset{\sim}{r},\underset{\sim}{\lambda}\right) = \left[\left(1 - m^2\right)rD_r + 2\right]L(r,m) = \overline{\vartheta u_p}\left[N_1(y) + m^2N_2(y)\right] \tag{126}$$

an equation that can only be satisfied provided that $L(r,m)$ is independent of m, in which case

$$L(y) = \frac{1}{2}\,\overline{\vartheta u_p}\left[N_1(y) + N_2(y)\right] \tag{127}$$

From these results, it follows that the correlation $\overline{\vartheta u_n{}'}$ (see fig. 2(b)) is given by

$$\frac{\overline{\vartheta u_n{}'}}{\overline{\vartheta u_p}} = m\sqrt{1 - m^2}N_2(y) = \frac{1}{2}\sin 2\alpha\ N_2(y) \tag{128}$$

The function $N_2(y)$ is plotted in figure 4 for $\sigma = 0$, 0.1, 0.71, 1, 10, and ∞.

Since the correlation of ϑ with the component of u_i orthogonal to the plane formed by $\underset{\sim}{r}$ and $\underset{\sim}{\lambda}$ vanishes by axisymmetry, it is necessary to consider only the correlation of ϑ with a velocity vector coplanar with $\underset{\sim}{r}$ and $\underset{\sim}{\lambda}$. The results given here for $\overline{\vartheta u_p{}'}$ and $\overline{\vartheta u_n{}'}$ then suffice to determine directly the correlation of ϑ with the velocity in an arbitrary direction without recourse to its scalar representation, such as those shown, for example, in figures 2(c) and 2(d).

The analytical and numerical results for $N(y,m)$ described above indicate that, although N decreases rather slowly for large values of y when σ is small, the Prandtl number dependence is not otherwise very strong. This is in marked contrast to the results for $M(y,m)$, especially those for large Prandtl numbers. In particular, although $M(y,m)$ decreases at a much more rapid rate than the velocity correlation $f(y)$ when the Prandtl number is large, $N(y,m)$ still decreases at a rate not significantly different from that for $f(y)$, just as it does for Prandtl numbers of order unity.

Mathematically, these conclusions are to some extent suggested by the forms of the basic equations (13) and (14) for the double temperature correlation θ and the temperature-velocity correlation L_i, respectively. In particular, equation (13) involves <u>only</u> the parameter κ, while equation (14) involves <u>both</u> κ and ν in the combination $\kappa + \nu$. The form of equation (14) then indicates that, when the Prandtl number is large and $\kappa + \nu \cong \nu$, L_i is approximately independent of the Prandtl number

and has a characteristic length scale essentially the same as that for R_{ij} (i.e., the microscale λ). On the other hand, for small Prandtl numbers, $\kappa + \nu \cong \kappa$, and L_i might then be expected to depend on the Prandtl number. Finally, equation (13) always involves κ explicitly and $\kappa + \nu$ implicitly (in the term involving L_i), so that θ might be expected to depend on κ and thus on the Prandtl number in general.

Physically, the temperature-velocity correlation is governed by both momentum transfer and heat transfer, with only the convective part of the latter (associated with the mean temperature gradient) being appreciable when the thermal conductivity is small. Thus, it is not too surprising that for small thermal conductivity this correlation behaves much like the velocity correlation. The double temperature correlation, on the other hand, depends only on heat transfer, and in this case the molecular part is always as important as the convective part. Consequently, the conclusions for the temperature correlation are also to some extent expected. Consider, for example, two fluids in turbulent motion with essentially the same dynamical and physical properties, except that one has low thermal conductivity (large Prandtl number) and the other high thermal conductivity (small Prandtl number). In order for the temperature fluctuations at two points to follow each other at all closely (and thus be significantly correlated), it would be expected that the two points would have to be rather close together in the first case but that they could be much farther apart in the second. It should be recalled, of course, that the present investigation is restricted to low Reynolds numbers, when molecular transport processes are the controlling factors. For high Reynolds numbers, the inertial transport processes neglected here are usually more important, and the final results may be quite different from the present ones.

Temperature and Heat-Transfer Spectrum Functions

In the last two sections the characteristics of the temperature field revealed by the one-point and two-point mean values have been studied. From some viewpoints, a further gain in physical insight is provided by an examination of the properties of the Fourier transforms of the correlation functions. These transforms and some functions closely related to them will now be considered in more detail.

Since the behavior of the general three-dimensional Fourier transform may in general be rather complicated, it is often convenient to define a simpler function by averaging over all directions of the vector argument $\underset{\sim}{k}$. Thus, by analogy with the definition of the energy-spectrum function for the velocity field, the three-dimensional (or spherically averaged) temperature-spectrum function may be defined as

$$\hat{\Theta}(k) = \int_{-1}^{1} \Theta(k,\mu)\, d\mu \tag{129}$$

This is just the integral of the Fourier transform $\Theta(k,\mu)\big/2\pi k^2$ of the correlation $\overline{\vartheta\vartheta'}$ over the surface of a sphere of radius k. Thus, $\hat{\Theta}$ is the density of contributions to $\overline{\vartheta^2}$ associated with wave numbers of magnitude k, so that

$$\overline{\vartheta^2} = \int_0^{\infty} \hat{\Theta}(k)\, dk \tag{130}$$

From equations (64) and (129) the following results are obtained:

$$\hat{\Theta}(k) = \frac{1}{3}\sqrt{\frac{2}{\pi}}\, \frac{\sigma^2}{(1 - \sigma)^2}\, R_\lambda{}^2\beta^2\lambda^3\left(e^{-x^2/2} - e^{-x^2/2\sigma}\right)^2 \tag{131}$$

where $x = k\sqrt{2\nu(t - t_o)}$. The maximum value of $\hat{\Theta}(k)$ is given by

$$\hat{\Theta}_m = \frac{1}{3}\sqrt{\frac{2}{\pi}}\, \sigma^{\frac{2}{1-\sigma}}\, R_\lambda{}^2\beta^2\lambda^3 \tag{132}$$

and occurs at

$$x_m = \left(\frac{2\sigma}{1 - \sigma}\log_e \frac{1}{\sigma}\right)^{1/2} \tag{133}$$

The function $\hat{\Theta}(k)/\Theta_m$ has been evaluated numerically for Prandtl numbers $\sigma = 0$, 0.01, 0.1, 0.71, 1, 10, 100, and ∞, and the results are shown in figure 5. The numerical values for the case $\sigma = 0.71$ are given in table 4. The wide range of variation of the results with the

Prandtl number is to be noted, as well as the dominance of the small wave-number contributions (associated with the large-scale structure of the temperature field) for small Prandtl numbers and the dominance of the large wave-number contributions (associated with the small-scale structure) for large Prandtl numbers. This behavior of the spectrum function is perhaps to be expected, on the basis of the previous conclusions for the correlation function.

The behavior $\hat{\Theta} \approx k^4$ for small values of the wave number, which is the same as it is for the turbulence-energy spectrum $E(k)$, is to be compared with Corrsin's result $\hat{\Theta} \approx k^2$ in the problem with constant mean temperature (ref. 16). It should be kept in mind that under consideration is the asymptotic part of the spectrum for large values of the time, which has characteristics determined mainly by the mean temperature gradient and the turbulence. Even in the present problem a part of the spectrum beginning with k^2 could be present, in general, as Mélèse (ref. 8) has shown (see eqs. (38) and (55)). However, as demonstrated previously, even if such a part were present initially, it would eventually make only a negligible contribution to the total spectrum function.

In a similar manner, a three-dimensional spectrum function $\hat{\Lambda}(k)$ is now defined for the mean turbulent heat transfer $-\overline{\vartheta u_p}$, where the negative sign is chosen in order to have a positive quantity (i.e., $-\overline{\vartheta u_p} = \left|\overline{\vartheta \underset{\sim}{u}}\right|$, as a result of the definition of $\underset{\sim}{\lambda}$):

$$\hat{\Lambda}(k) = -2\pi k^2 \int_{-1}^{1} \Lambda_p(k,\mu)\ d\mu \tag{134}$$

Thus, $\hat{\Lambda}(k)$ is the density of contributions to $-\overline{\vartheta u_p}$ associated with wave numbers of magnitude k, and

$$-\overline{\vartheta u_p} = \int_0^\infty \hat{\Lambda}(k)\ dk \tag{135}$$

Equations (30), (63), and (134) then give the results

$$\hat{\Lambda}(k) = -\frac{8\pi}{3} k^2 \Lambda(k)$$

$$= \frac{2}{3}\sqrt{\frac{2}{\pi}} \frac{\sigma}{1 - \sigma} R_\lambda \left(\overline{u^2}\right)^{1/2} \beta\lambda^2 x^2 \left(e^{-x^2} - e^{-\frac{1+\sigma}{2\sigma}x^2}\right) \tag{136}$$

The maximum value Λ_m occurs at a value of x given by

$$x_m^2 = \frac{1 - e^{-\frac{1-\sigma}{2\sigma}x_m^2}}{1 - \frac{1 + \sigma}{2\sigma} e^{-\frac{1-\sigma}{2\sigma}x_m^2}} \tag{137}$$

The function $\hat{\Lambda}(k)/\hat{\Lambda}_m$ has been evaluated numerically for Prandtl numbers $\sigma = 0$, 0.1, 0.71, 1, 10, and ∞, and the results are shown in figure 6. The numerical values for the case $\sigma = 0.71$ are given in table 4. In contrast to the conclusions for $\hat{\Theta}(k)/\hat{\Theta}_m$, the variation of $\hat{\Lambda}(k)/\hat{\Lambda}_m$ with the Prandtl number is not large. In fact, the main contributions always tend to come from essentially the same range of wave numbers as that corresponding to the energy-containing eddies of the turbulence.

Equation (136) shows that $\hat{\Lambda}(k)$ begins with a term of order k^4 for small values of k during the final period of decay, just as it does, in general, initially (see eqs. (36) and (56)). The general conclusion that the heat-transfer spectrum begins with at least the fourth power of k was obtained previously by Mélèse (ref. 8).

Although the three-dimensional spectrum, as described in the preceding two cases, has perhaps the greatest physical significance, it cannot be directly measured. One-dimensional spectrum functions are more suitable for experimental purposes, since they can be obtained fairly easily from correlation measurements and in fact can sometimes be measured directly. Some useful results relating such one-dimensional spectrum functions with the three-dimensional ones are now presented.

A one-dimensional Fourier transform of the double temperature correlation $\theta\left(\underset{\sim}{r},\underset{\sim}{\lambda}\right) = \overline{\vartheta\vartheta'}$ can be made along a line at an angle α_1 with $\underset{\sim}{\lambda}$ as follows:

$$\Theta_I\left(k_1,m_1\right) = (2\pi)^{-1}\int_{-\infty}^{\infty} \theta\left(r_1,0,0\right)e^{-ik_1r_1}\,dr_1 \tag{138}$$

where the dependence on $m_1 = \cos\alpha_1$ due to the axisymmetry of the temperature field has been explicitly indicated. Since $\theta\left(\underset{\sim}{r},\underset{\sim}{\lambda}\right) = \theta(r,m)$ is an even function of both r and m, equation (138) can be written as

$$\Theta_I\left(k_1,m_1\right) = \frac{1}{\pi}\int_0^{\infty} \theta\left(r_1,m_1\right)\cos\left(k_1r_1\right)dr_1 \tag{139}$$

Then, with the aid of equation (95), this becomes

$$\Theta_I\left(k_1,m_1\right) = \frac{2}{\pi}\int_0^{\infty}\int_0^{\infty}\int_0^{1} \Theta(k,\mu)\cos\left(k_1r_1\right)\cos\left(kr_1m_1\mu\right)J_0\left[kr_1\sqrt{\left(1 - m_1^2\right)\left(1 - \mu^2\right)}\right]d\mu\,dk\,dr_1 \tag{140}$$

After substitution for $\Theta(k,\mu)$ in terms of $\hat{\Theta}(k)$ from equation (131), the integration with respect to μ and r_1 can be easily carried out in the cases $m_1 = 0$ and $m_1 = 1$ (see ref. 15, p. 43), with the results

$$\Theta_I\left(k_1,0\right) = \frac{3}{8}\int_{k_1}^{\infty}\frac{\hat{\Theta}(k)}{k}\left(1 + \frac{k_1^2}{k^2}\right)dk \tag{141}$$

$$\Theta_I\left(k_1,1\right) = \frac{3}{4}\int_{k_1}^{\infty}\frac{\hat{\Theta}(k)}{k}\left(1 - \frac{k_1^2}{k^2}\right)dk \tag{142}$$

Since $\theta(r_1,m_1) = \theta(r_1,0) + m_1^2\left[\theta(r_1,1) - \theta(r_1,0)\right]$, the general formula is seen to be

$$\Theta_I(k_1,m_1) = \left(1 - m_1^2\right)\Theta_I(k_1,0) + m_1^2\Theta_I(k_1,1) \tag{143}$$

The inverse relations to these are as follows:

$$\hat{\Theta}(k) = -\frac{4}{3}k\left[\Theta_I'(k,0) - \int_0^k \frac{\Theta_I'(k,0)}{k}\,dk\right] \tag{144}$$

$$\hat{\Theta}(k) = \frac{2}{3}k^3\frac{d}{dk}\left[\frac{\Theta_I'(k,1)}{k}\right] \tag{145}$$

$$\hat{\Theta}(k) = -\frac{4k}{3\left(1 - m_1^2\right)}\left[\Theta_I'\left(k,m_1\right) - \frac{1 - 3m_1^2}{1 - m_1^2}k^{\frac{2m_1^2}{1-m_1^2}}\int_0^k k^{\frac{m_1^2+1}{m_1^2-1}}\Theta_I'\left(k,m_1\right)dk\right] \tag{146}$$

The prime denotes differentiation with respect to k. The integral from 0 to k in relations (144) and (146) can be replaced by the integral from ∞ to k, since it can be shown that the integral from 0 to ∞ is zero in each case. From any one of these relations, then, the three-dimensional temperature spectrum can be determined from a measured one-dimensional spectrum. It appears that formula (144) should be the most useful since $\Theta_I(k_1,0)$ is a Fourier transform in a direction normal to $\underset{\sim}{\lambda}$. In an experimental study of the present problem, the mean flow direction would likely be normal to $\underset{\sim}{\lambda}$, so that a direct measurement of $\Theta_I(k_1,0)$ would be possible (subject to certain approximations), without having to determine it from a correlation measurement.

It might be mentioned that in the limit $\sigma \rightarrow 0$ the one-dimensional spectrum function Θ_I is well behaved for small values of k, in spite of an apparent logarithmic divergence of the integrals in equations (141) and (142) in the limiting case. Since $\hat{\Theta}$ has a maximum value of the order of σ^2 at a value of x of the order of $(\sigma \log_e \sigma)^{1/2}$ (see

equations (132) and (133)), it follows from equations (141) and (142) that Θ_I has a maximum value of the order of $\sigma^2 \log_e{}^2 \sigma$. Although this is much larger than the maximum value of $\hat{\Theta}$, it still approaches zero in the limit $\sigma \to 0$.

For the temperature-velocity correlation $L_i(\underset{\sim}{r}, \underset{\sim}{\lambda}) = \overline{\vartheta u_i{}'}$, since it is a vector quantity, many one-dimensional spectrum functions could be defined. Two of the most useful ones appear to be the following:

$$\begin{aligned} \Lambda_I(k_1, m_1) &= (2\pi)^{-1} \int_{-\infty}^{\infty} L_p(r_1, 0, 0) e^{-ik_1 r_1} \, dr_1 \\ &= \frac{1}{\pi} \int_0^{\infty} L_p(r_1, m_1) \cos (k_1 r_1) \, dr_1 \end{aligned} \tag{147}$$

$$\begin{aligned} \Lambda_{II}(k_1, m_1) &= (2\pi)^{-1} \int_{-\infty}^{\infty} L_n(r_1, 0, 0) e^{-ik_1 r_1} \, dr_1 \\ &= \frac{1}{\pi} \int_0^{\infty} L_n(r_1, m_1) \cos (k_1 r_1) \, dr_1 \end{aligned} \tag{148}$$

With the aid of equations (110) and (136), the following special results can be obtained in the manner described previously:

$$\Lambda_I(k_1, 0) = -\frac{3}{8} \int_{k_1}^{\infty} \frac{\hat{\Lambda}(k)}{k} \left(1 + \frac{k_1{}^2}{k^2}\right) dk \tag{149}$$

$$\Lambda_I(k_1, 1) = -\frac{3}{4} \int_{k_1}^{\infty} \frac{\hat{\Lambda}(k)}{k} \left(1 - \frac{k_1{}^2}{k^2}\right) dk \tag{150}$$

From these, since $L_p(r_1, m_1) = L_p(r_1, 0) + m_1{}^2 \left[L_p(r_1, 1) - L_p(r_1, 0)\right]$, the general case follows:

$$\Lambda_I(k_1,m_1) = \left(1 - m_1^2\right)\Lambda_I(k_1,0) + m_1^2\Lambda_I(k_1,1) \tag{151}$$

Also, since $L_n(r_1,m_1) = m_1\sqrt{1 - m_1^2}\left[L_p(r_1,1) - L_p(r_1,0)\right]$, (see fig. 2(b)), the result for Λ_{II} is

$$\Lambda_{II}(k_1,m_1) = m_1\sqrt{1 - m_1^2}\left[\Lambda_I(k_1,1) - \Lambda_I(k_1,0)\right] \tag{152}$$

The inverse relations to relations (149), (150), and (152) are

$$\hat{\Lambda}(k) = \frac{4}{3} k\left[\Lambda_I'(k,0) - \int_0^k \frac{\Lambda_I'(k,0)}{k} dk\right] \tag{153}$$

$$\hat{\Lambda}(k) = -\frac{2}{3} k^3 \frac{d}{dk}\left[\frac{\Lambda_I'(k,1)}{k}\right] \tag{154}$$

$$\hat{\Lambda}(k) = -\frac{4k}{3m_1\sqrt{1 - m_1^2}}\left[\Lambda_{II}'(k,m_1) - \frac{3}{k^2}\int_0^k k\Lambda_{II}'(k,m_1)\, dk\right] \tag{155}$$

The integrals from 0 to k may again be replaced by integrals from ∞ to k.

Although the one-dimensional spectrum $\Lambda_I(k_1,0)$ could perhaps be measured directly in the experimental situation, such a determination is not so easy as that for $\Theta_I(k_1,0)$, and a more satisfactory procedure appears to be to obtain a one-dimensional spectrum from a measured correlation, such as $L_n(r_1,m_1)$, for example. From the corresponding one-dimensional spectrum $\Lambda_{II}(k_1,m_1)$, the three-dimensional heat-transfer spectrum $\hat{\Lambda}(k)$ could then be obtained by means of equation (155).

It should be emphasized that all of the special relations between the various spectrum functions derived above are valid only under the special restrictions of the present investigation.

CONCLUDING REMARKS

In the present investigation of heat transfer in isotropic turbulence with a constant mean temperature gradient a variety of special results have been derived for the final period of decay. From these results the following major conclusions emerge:

(a) During the final period of decay, the temperature field becomes asymptotically independent of the initial conditions on the temperature. Its characteristics eventually depend essentially only on the mean temperature gradient, the physical properties of the fluid, and the characteristics of the turbulence.

(b) For a given field of turbulence, all results depending on temperature fluctuations alone are strongly dependent on the Prandtl number. Examples are the results for the mean square $\overline{\vartheta^2}$ of the temperature fluctuations and for the double-temperature-correlation coefficient $\overline{\vartheta\vartheta'}/\overline{\vartheta^2}$. It is particularly noteworthy that for small Prandtl numbers the temperature fluctuations remain significantly correlated over much larger distances than the velocity fluctuations, while for large Prandtl numbers the opposite is true.

(c) Many of the results depending on both the temperature and velocity fluctuations display much less dependence on the Prandtl number. For example, the mean turbulence heat transfer $\overline{\vartheta u_p}$ is of the same order of magnitude for very large Prandtl numbers as it is for a Prandtl number of unity. The Prandtl number variation of such quantities is usually more important for small Prandtl numbers. The quantity $\overline{\vartheta u_p}$, for example, approaches zero when the Prandtl number approaches zero. However, the behavior of the temperature-velocity correlation coefficient $\overline{\vartheta u_p'}/\overline{\vartheta u_p}$ changes relatively little over the full range of Prandtl numbers, in marked contrast to the behavior of the double temperature correlation coefficient.

(d) Corresponding trends are present in the behavior of the three-dimensional spectrum functions. For the temperature spectrum, the small wave-number part is most important when the Prandtl number is small, while the large wave-number part is most important when the Prandtl number is large. For the heat-transfer spectrum, however, the main contributions

always come from essentially the same range of wave numbers as that corresponding to the energy-containing eddies of the turbulence.

In evaluating these conclusions, the restriction to small Reynolds and Péclet numbers should be kept in mind; however, although the results are strictly valid only in the limit of vanishing R_λ and P_{λ_ϑ}, any radical changes for small but finite values of these parameters would not be expected. Experience with other such approximations suggests that the present results might even be quantitatively accurate for values of these parameters up to 1 and perhaps qualitatively correct up to values of about 5. It would, of course, be desirable to investigate the deviation from the limiting results for zero R_λ and P_{λ_ϑ} by means of a second-order approximation, in which terms of order R_λ and P_{λ_ϑ} are retained. In such an analysis, the triple correlations, which are neglected in the present study, would have to be considered. Thus, in addition to corrections to the present results, conclusions regarding the role of the triple correlations for small Reynolds and Péclet numbers, which are of great interest in themselves, would also be obtained.

The situation at high Reynolds and Péclet numbers, it should be emphasized, is likely to be quite different from that described by the present theory or any extension of it in the manner just outlined. In these circumstances, the inertial transfer terms in the basic equations, which should be small for the present type of analysis to be valid, are normally dominant. Consequently, a different mathematical approach has to be used, and the results would be expected to be qualitatively quite different. For example, one would not expect to find a strong Prandtl number dependence of the properties associated with the large-scale structure of the temperature and velocity fields, as found here in some cases, although such a dependence would likely still be present in the characteristics of the small-scale structure. In an investigation of this situation, the statistical hypothesis mentioned in the introduction appears to be essential in formulating a mathematical problem that is at all tractable. As suggested previously, there are still serious analytical difficulties, but it may be expected that eventually this approach will lead to a solution of the problem.

The Johns Hopkins University,
Baltimore, Md., February 25, 1957,
and
Ballistic Research Laboratories, U. S. Army,
Aberdeen Proving Ground, Md., February 25, 1957.

REFERENCES

1. Corrsin, Stanley: Heat Transfer in Isotropic Turbulence. Jour. Appl. Phys., vol. 23, no. 1, Jan. 1952, pp. 113-118.

2. Reiss, F. B.: Studies of Correlation and Spectra in Homogeneous Turbulence. Ph. D. Thesis, M.I.T., 1952.

3. Burgers, J. M., and Mitchner, M.: On Homogeneous Non-Isotropic Turbulence Connected With a Mean Motion Having a Constant Velocity Gradient, Proc. Koninklijke Nederlandse Akademie van Wetenschappen (Netherlands). I. Ser. B, vol. 56, no. 3, 1953, pp. 228-235. II. Ser. B, vol. 56, no. 4, 1953, pp. 343-354.

4. Proudman, I., and Reid, W. H.: On the Decay of a Normally Distributed and Homogeneous Turbulent Velocity Field. Phil. Trans. Roy. Soc. (London), ser. A, vol. 247, Nov. 4, 1954, pp. 163-189.

5. Tatsumi, T.: The Theory of Decay Process of Incompressible Isotropic Turbulence. Proc. Roy. Soc. (London), ser. A, vol. 239, no. 1216, Feb. 12, 1957, pp. 16-45.

6. Reid, W. H.: On the Stretching of Material Lines and Surfaces in Isotropic Turbulence With Zero Fourth Cumulants. Proc. Cambridge Phil. Soc., vol. 51, pt. 2, Apr. 1955, pp. 350-362.

7. Chandrasekhar, S.: A Theory of Turbulence. Proc. Roy. Soc. (London), ser. A, vol. 229, no. 1176, Apr. 5, 1955, pp. 1-19.

8. Mélèse, G. B.: On Axisymmetric Turbulence and Scalar Fields. Ph. D. Diss., The Johns Hopkins Univ., 1954.

9. Corrsin, Stanley: The Decay of Isotropic Temperature Fluctuations in an Isotropic Turbulence. Jour. Aero. Sci., vol. 18, no. 6, June, 1951, pp. 417-423.

10. Chandrasekhar, S.: On Turbulence Caused by Thermal Instability. Phil. Trans. Roy. Soc. (London), ser. A, vol. 244, no. 884, Mar. 11, 1952, pp. 357-384.

11. Chandrasekhar, S.: The Theory of Axisymmetric Turbulence. Phil. Trans. Roy. Soc. (London), ser. A, vol. 242, no. 855, Sept. 5, 1950, pp. 557-577.

12. Batchelor, G. K., and Proudman, I.: The Large-Scale Structure of Homogeneous Turbulence. Phil. Trans. Roy. Soc. (London), ser. A, vol. 248, no. 949, Jan. 5, 1956, pp. 369-405.

13. Sommerfeld, A.: Partial Differential Equations in Physics. Academic Press, Inc. (New York), 1949.

14. Batchelor, G. K.: The Theory of Homogeneous Turbulence. Cambridge Univ. Press, 1953.

15. Staff of the Bateman Manuscript Project, A. Erdélyi, ed.: Tables of Integral Transforms. Vol. I. McGraw-Hill Book Co., Inc., 1954.

16. Corrsin, S.: On the Spectrum of Isotropic Temperature Fluctuations in an Isotropic Turbulence. Jour. Appl. Phys., vol. 22, no. 4, Apr. 1951, pp. 469-473.

TABLE 1.- FUNCTIONS DESCRIBING TEMPERATURE

CORRELATION $M(y,m) = M_1(y) + m^2M_2(y)$

FOR PRANDTL NUMBER $\sigma = 0$

y	$M_1(y)$	$M_2(y)$	$M_1(y) + M_2(y)$
0	1.0000	0	1.0000
.2	.9960	.0020	.9980
.4	.9842	.0079	.9921
.6	.9650	.0173	.9823
.8	.9392	.0299	.9691
1.0	.9076	.0450	.9526
1.2	.8713	.0619	.9332
1.4	.8315	.0800	.9114
1.6	.7893	.0984	.8877
1.8	.7459	.1165	.8624
2.0	.7022	.1338	.8360
2.2	.6592	.1498	.8090
2.4	.6176	.1641	.7817
2.6	.5779	.1765	.7544
2.8	.5405	.1870	.7275
3.0	.5057	.1954	.7011
3.2	.4735	.2019	.6754
3.4	.4440	.2066	.6506
3.6	.4171	.2097	.6268
3.8	.3926	.2114	.6040
4.0	.3704	.2119	.5823
4.2	.3503	.2113	.5616
4.4	.3321	.2099	.5420
4.6	.3156	.2079	.5234
4.8	.3006	.2053	.5059
5.0	.2869	.2023	.4892
5.2	.2744	.1991	.4735
5.4	.2630	.1956	.4586
5.6	.2525	.1920	.4445
5.8	.2428	.1883	.4311
6.0	.2339	.1846	.4185
6.5	.2142	.1755	.3897
7.0	.1977	.1667	.3643
7.5	.1835	.1583	.3419
8.0	.1714	.1506	.3219
8.5	.1607	.1434	.3041
9.0	.1514	.1368	.2881
9.5	.1430	.1306	.2737
10.0	.1356	.1250	.2606

TABLE 2.- FUNCTIONS DESCRIBING TEMPERATURE-VELOCITY

CORRELATION $N(y,m) = N_1(y) + m^2N_2(y)$

FOR PRANDTL NUMBER $\sigma = 0$

y	$N_1(y)$	$N_2(y)$	$N_1(y) + N_2(y)$
0	1.0000	0	1.0000
.2	.9881	.0058	.9939
.4	.9530	.0233	.9763
.6	.8970	.0507	.9477
.8	.8236	.0857	.9093
1.0	.7369	.1256	.8626
1.2	.6418	.1675	.8094
1.4	.5432	.2083	.7515
1.6	.4455	.2455	.6909
1.8	.3526	.2768	.6294
2.0	.2676	.3008	.5684
2.2	.1926	.3169	.5094
2.4	.1286	.3249	.4535
2.6	.0761	.3253	.4014
2.8	.0345	.3190	.3535
3.0	.0030	.3073	.3103
3.2	-.0198	.2914	.2716
3.4	-.0353	.2727	.2373
3.6	-.0449	.2522	.2071
3.8	-.0500	.2312	.1812
4.0	-.0518	.2103	.1585
4.2	-.0513	.1902	.1390
4.4	-.0492	.1714	.1222
4.6	-.0463	.1540	.1077
4.8	-.0429	.1382	.0953
5.0	-.0394	.1240	.0846
5.2	-.0359	.1113	.0754
5.4	-.0327	.1001	.0674
5.6	-.0296	.0901	.0605
5.8	-.0269	.0814	.0545
6.0	-.0244	.0736	.0492
6.5	-.0193	.0580	.0387
7.0	-.0155	.0465	.0310
7.5	-.0126	.0378	.0252
8.0	-.0104	.0312	.0208
8.5	-.0087	.0260	.0173
9.0	-.0073	.0219	.0146
9.5	-.0062	.0186	.0125
10.0	-.0053	.0160	.0106

TABLE 3.- FUNCTIONS DESCRIBING TEMPERATURE CORRELATION

$M(y,m) = M_1(y) + m^2M_2(y)$ AND TEMPERATURE-VELOCITY

CORRELATION $N(y) = N_1(y) + m^2N_2(y)$ FOR

PRANDTL NUMBER $\sigma = 0.71$

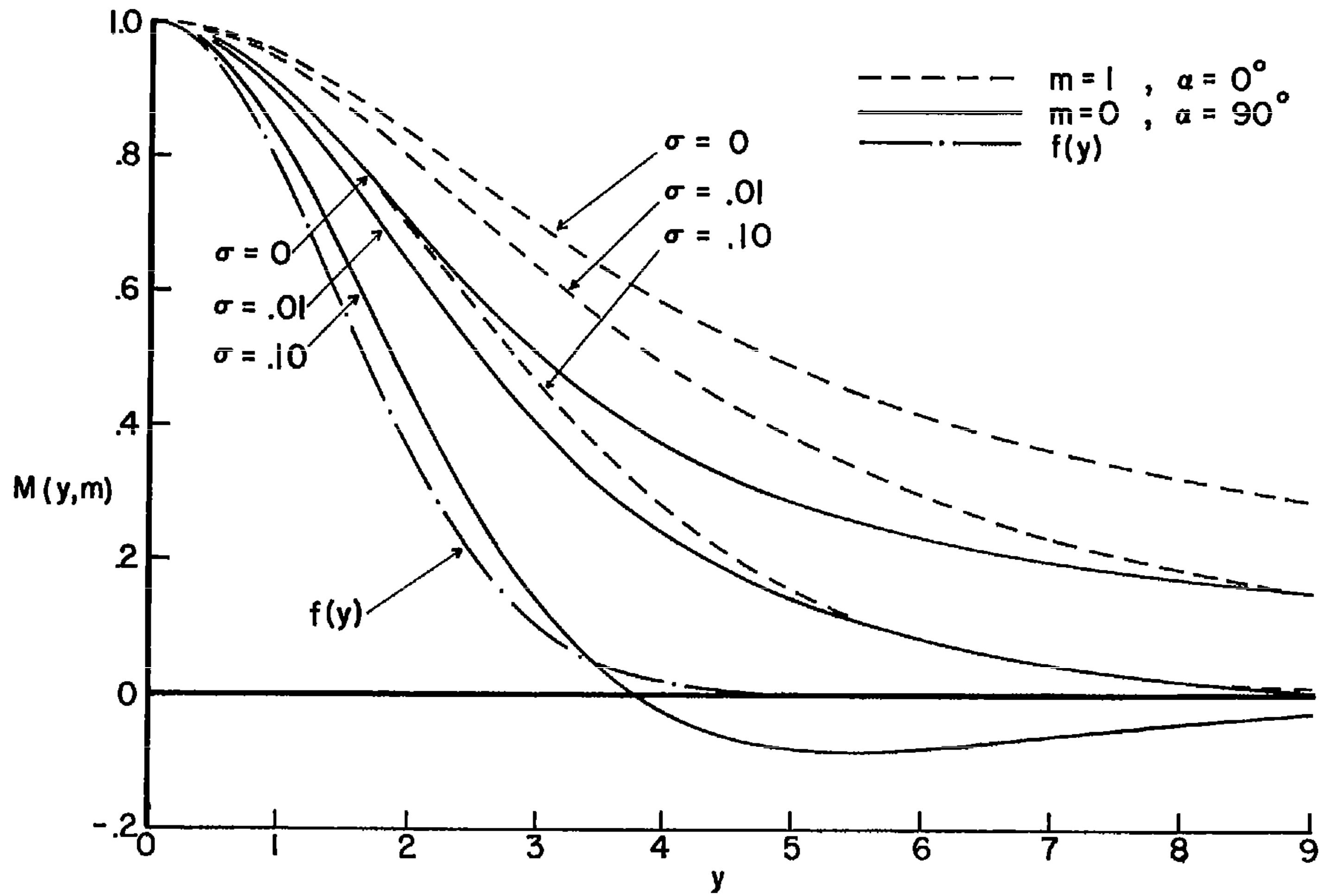
y	M_1	M_2	$M_1 + M_2$	N_1	N_2	$N_1 + N_2$
0	1.000	0	1.000	1.000	0	1.000
.4	.934	.033	.967	.929	.035	.964
.8	.755	.115	.870	.737	.126	.863
1.2	.514	.218	.732	.482	.237	.719
1.6	.270	.309	.579	.231	.326	.557
2.0	.071	.359	.430	.034	.366	.400
2.4	-.060	.358	.298	-.085	.352	.267
2.8	-.124	.317	.193	-.131	.298	.167
3.2	-.133	.248	.115	-.128	.225	.097
3.6	-.110	.178	.068	-.101	.153	.052
4.0	-.084	.120	.036	-.069	.095	.026
5.0	-.024	.024	.000	-.016	.019	.003

TABLE 4.- TEMPERATURE AND HEAT-TRANSFER SPECTRUM

FUNCTIONS $\hat{\Theta}(x)/\hat{\Theta}_m$ AND $\hat{\Lambda}(x)/\hat{\Lambda}_m$ FOR

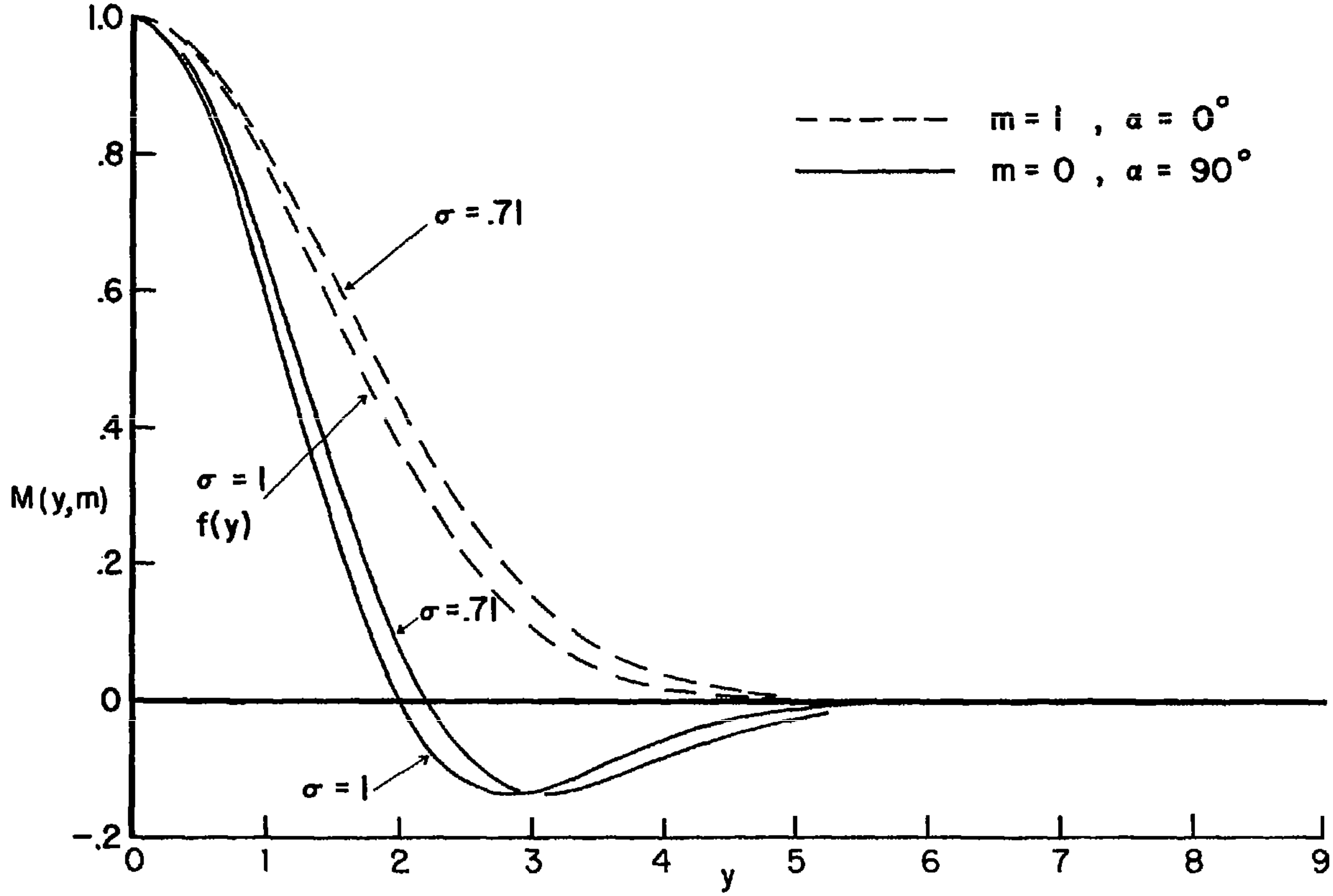
PRANDTL NUMBER $\sigma = 0.71$

x	$\hat{\Theta}/\hat{\Theta}_m$	$\hat{\Lambda}/\hat{\Lambda}_m$
0	0	0
.2	.0041	.0034
.4	.0561	.0479
.6	.2229	.1944
.8	.5035	.4516
1.0	.7983	.7422
1.2	.9784	.9496
1.4	.9751	.9947
1.6	.8151	.8801
1.8	.5837	.6709
2.0	.3631	.4468
2.2	.1983	.2625
2.4	.0959	.1371
2.8	.0160	.0269
3.2	.0017	.0035
3.6	.0001	.0003
4.0	.0000	.0000



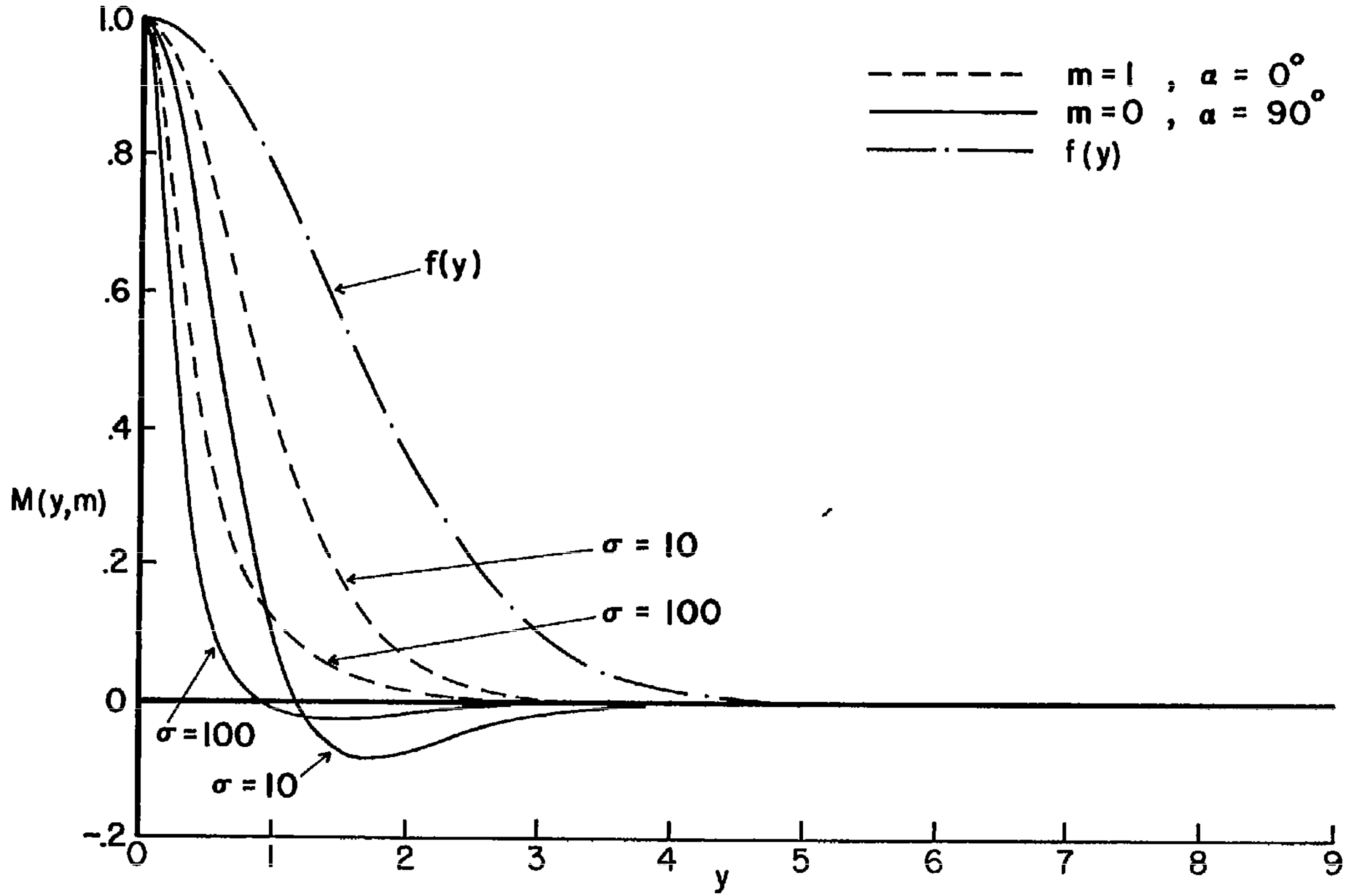
(a) Prandtl numbers $\sigma = 0$, $\sigma = 0.01$, and $\sigma = 0.10$.

Figure 1.- Temperature correlation coefficient $M(y,m) = \overline{\theta\theta'}/\overline{\theta^2}$. $m = \cos\alpha$; $y = \sqrt{2}\, r/\lambda$; λ = Turbulence microscale; $f(y)$ = Velocity correlation.



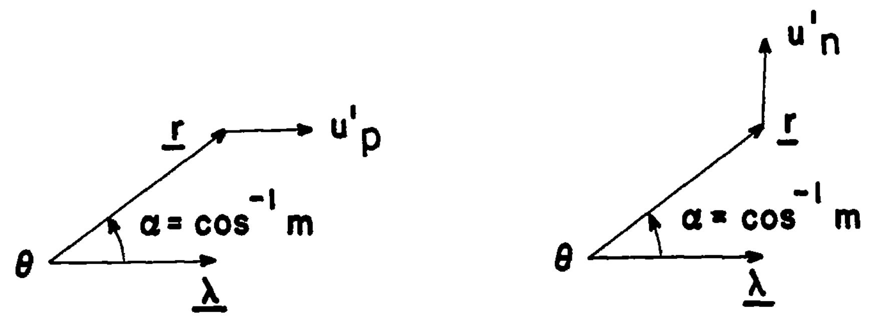
(b) Prandtl numbers σ = 0.71 and σ = 1.

Figure 1.- Continued.

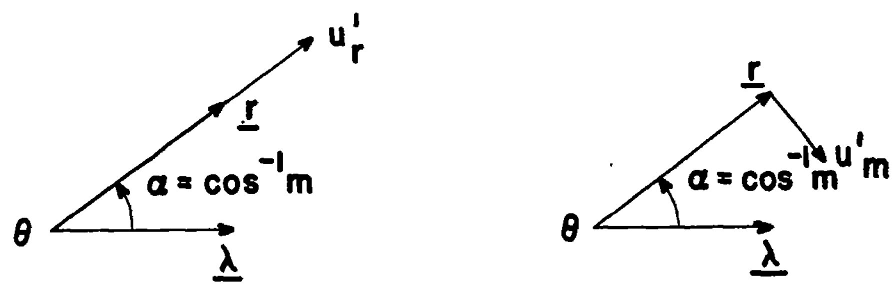


(c) Prandtl numbers σ = 10 and σ = 100.

Figure 1.- Concluded.

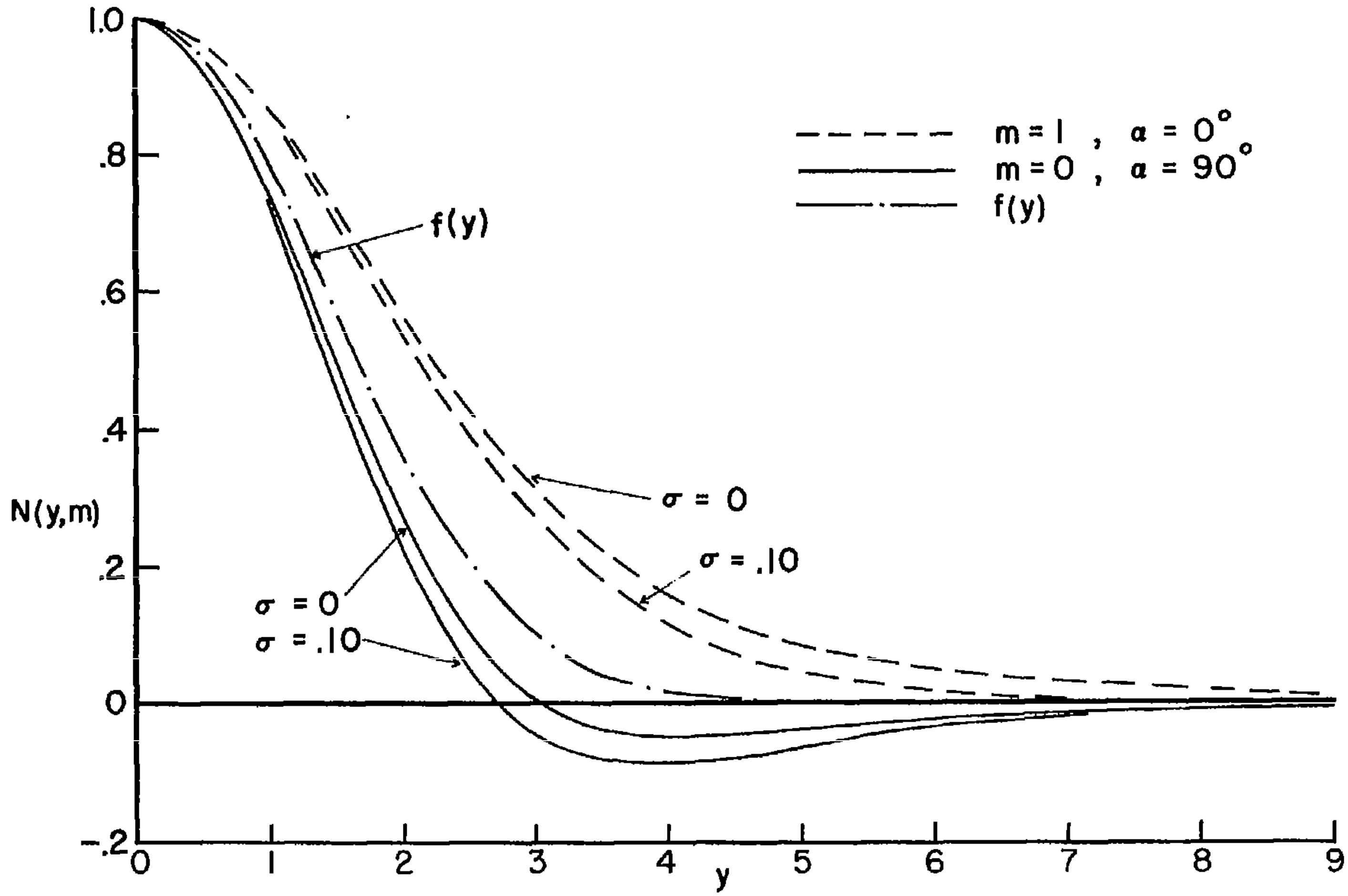


(a) $\dfrac{\overline{\theta u'_p}}{\overline{\theta u_p}} = N_1(y) + m^2 N_2(y).$ (b) $\dfrac{\overline{\theta u'_n}}{\overline{\theta u_p}} = \frac{1}{2} \sin 2\alpha \; N_2(y).$



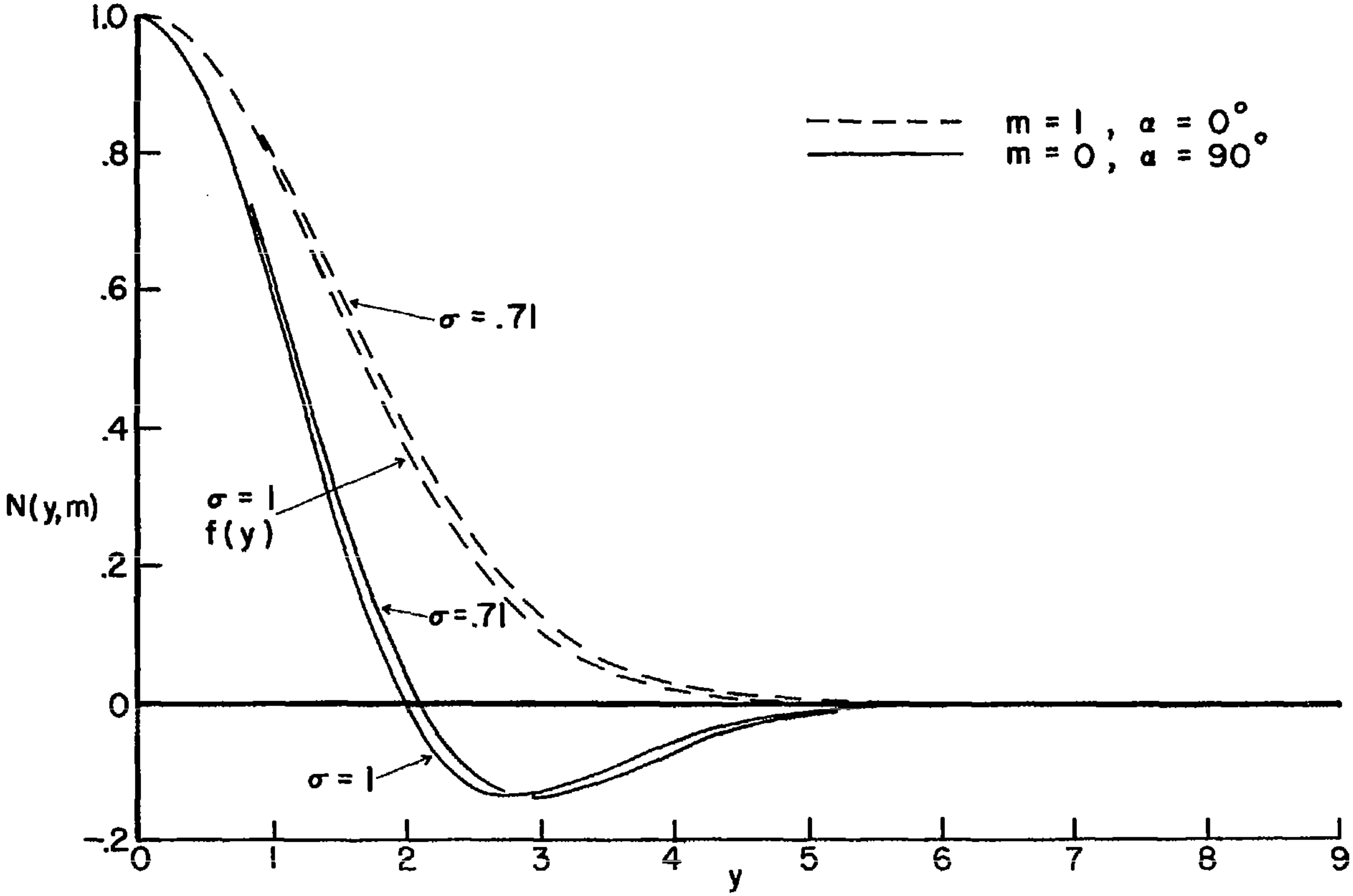
(c) $\dfrac{\overline{\theta u'_r}}{\overline{\theta u_p}} = \cos \alpha \left[N_1(y) + N_2(y)\right].$ (d) $\dfrac{\overline{\theta u'_m}}{\overline{\theta u_p}} = \sin \alpha \; N_1(y).$

Figure 2.- Temperature-velocity correlations.



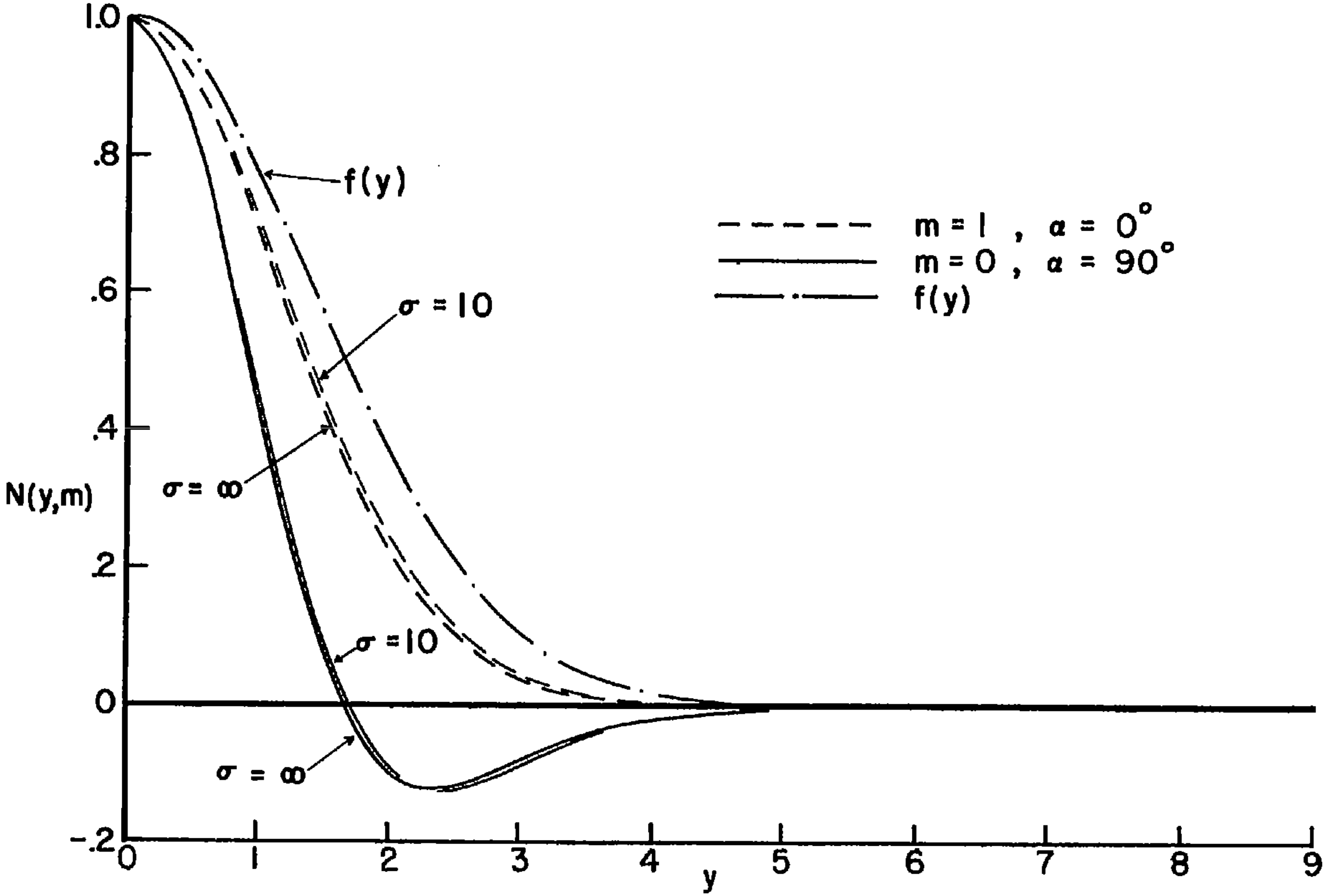
(a) Prandtl numbers $\sigma = 0$ and $\sigma = 0.10$.

Figure 3.- Temperature-velocity correlation coefficient $N(y,m) = \overline{\theta u'_p}/\overline{\theta u_p}$. $m = \cos\alpha$; $y = \sqrt{2}\, r/\lambda$; λ = Turbulence microscale; $f(y)$ = Velocity correlation.



(b) Prandtl numbers σ = 0.71 and σ = 1.

Figure 3.- Continued.



(c) Prandtl numbers $\sigma = 10$ and $\sigma = \infty$.

Figure 3.- Concluded.

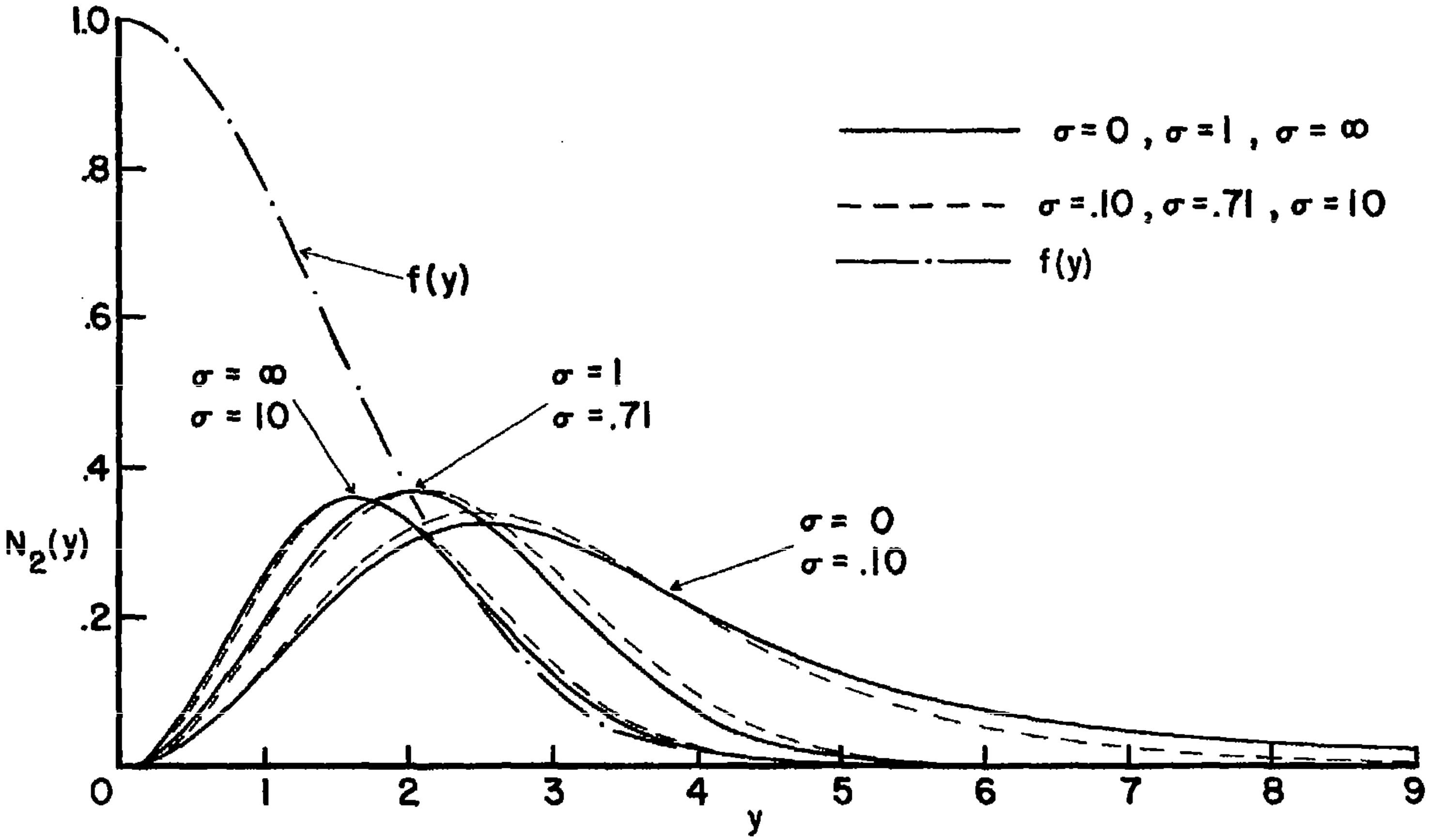


Figure 4.- Temperature-velocity correlation coefficient $N_2(y) = (2/\sin 2\alpha)\overline{\theta u'_n}/\overline{\theta u_p}$ for various Prandtl numbers. $f(y)$ = Velocity correlation; $y = \sqrt{2}\, r/\lambda$; λ = Turbulence microscale.

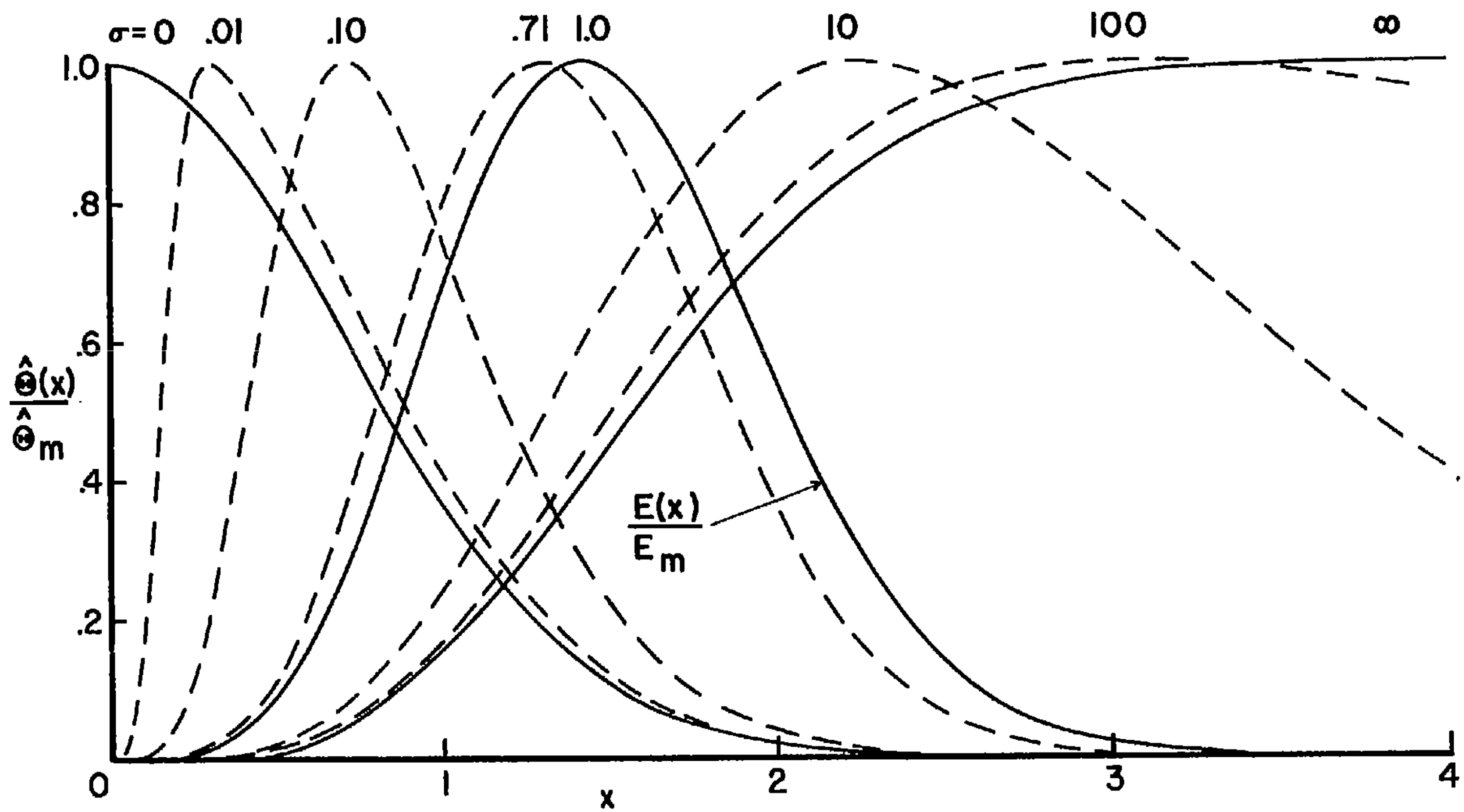


Figure 5.- Three-dimensional temperature-spectrum function $\hat{\theta}(x)$ for various Prandtl numbers. $x = k\lambda/\sqrt{2}$; $\hat{\theta}_m$ = Maximum value of $\hat{\theta}$; λ = Turbulence microscale; $E(x)$ = Turbulence energy spectrum.

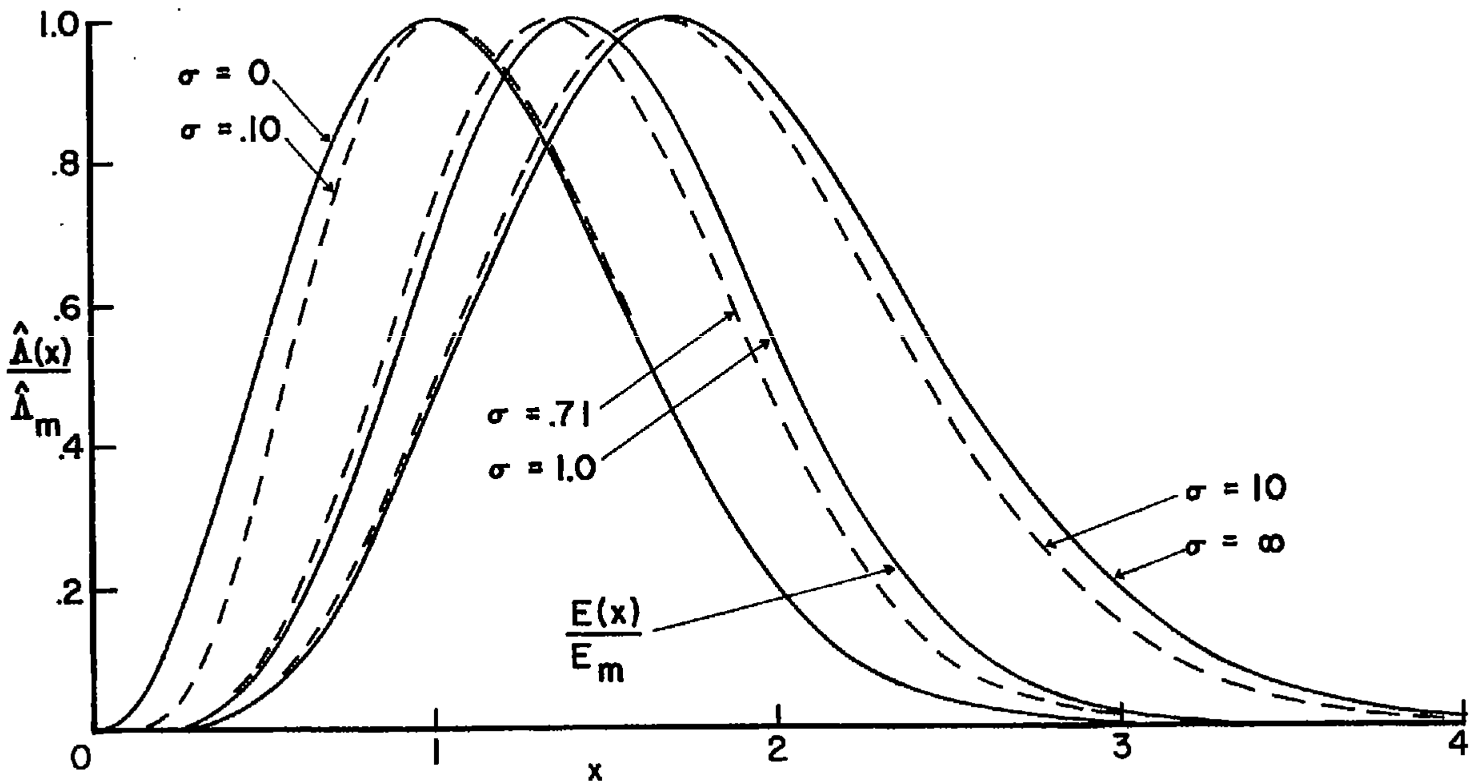


Figure 6.- Three-dimensional heat-transfer spectrum function $\hat{\Lambda}(x)$ for various Prandtl numbers. $x = k\lambda/\sqrt{2}$; $\hat{\Lambda}_m$ = Maximum value of $\hat{\Lambda}$; λ = Turbulence microscale; $E(x)$ = Turbulence energy spectrum.